The Accuracy Cost of Weakness: A Theoretical Analysis of Fixed-Segment Weak Labeling for Events in Time

John Martinsson *john.martinsson@ri.se*
Computer Science
RISE Research Institutes of Sweden
Centre for Mathematical Sciences
Lund University

Olof Mogren *olof.mogren@ri.se*
Computer Science
RISE Research Institutes of Sweden

Tuomas Virtanen *tuomas.virtanen@tuni.fi*
Signal Processing Research Centre
Tampere University

Maria Sandsten *maria.sandsten@matstat.lu.se*
Centre for Mathematical Sciences
Lund University

Abstract

Accurate labels are critical for deriving robust machine learning models. Labels are used to train supervised learning models and to evaluate most machine learning paradigms. In this paper, we model the accuracy and cost of a common weak labeling process where annotators assign presence or absence labels to fixed-length data segments for a given event class. The annotator labels a segment as "present" if it sufficiently covers an event from that class, e.g., a birdsong sound event in audio data. We analyze how the segment length affects the label accuracy and the required number of annotations, and compare this fixed-length labeling approach with an oracle method that uses the true event activations to construct the segments. Furthermore, we quantify the gap between these methods and verify that in most realistic scenarios the oracle method is better than the fixed-length labeling method in both accuracy and cost. Our findings provide a theoretical justification for adaptive weak labeling strategies that mimic the oracle process, and a foundation for optimizing weak labeling processes in sequence labeling tasks.

1 Introduction

In supervised machine learning, labeled datasets are required for training and evaluation. During evaluation, the accuracy of the labels determine the quality of the analysis. However, in practice, labels often contain noise that varies with the input sample and label type. Noisy training labels present a persistent challenge in machine learning (Liang et al., 2009; Song et al., 2022). Deep learning models, in particular, are prone to overfitting noisy labels, raising questions about the nature of generalization (Zhang et al., 2021). Regularization techniques such as dropout (Srivastava et al., 2014), data augmentation (Shorten & Khoshgoftaar, 2019), and weight decay (Krogh & Hertz, 1991) mitigate overfitting but fail to eliminate the performance gap between training on noisy versus clean labels (Song et al., 2022).

Beyond the well-documented challenges posed by noisy training labels, inaccurate evaluation labels present a significant, yet often overlooked, obstacle to reliable machine learning. When evaluation metrics are computed against noisy ground truth, the apparent "best" performing model might simply be the one that most closely reproduces the noise present in the evaluation set, rather than exhibiting superior generalization capabilities. This very issue, where noisy evaluation labels can lead to the rejection of models that have learned the true clean label distribution, is a central concern addressed by Görnitz et al. (2014). This can lead to the selection of suboptimal models that perform well on the flawed evaluation data but generalize poorly to unseen, cleaner data or data from real-world applications. Consequently, performance benchmarks can be inflated and misleading, hindering meaningful comparisons between different approaches. Therefore, understanding the characteristics of label noise, not just in the training data but also in the evaluation data, is crucial for developing and selecting models that are truly effective and robust.

Labels are typically obtained through human annotation, a process that involves significant time and financial investment, particularly for complex data like audio or time-series signals. In this work, we consider a form of weak labeling where the annotator assigns presence or absence labels to predefined data segments. This offers a practical and cost-effective approach for annotating large audio datasets (Martin-Morato & Mesaros, 2023). To reduce cost, weak labels avoid specifying precise boundaries within the data segments, focusing instead on general presence or absence of the target class. However, this simplification introduces noise into the labels, especially for data with time-varying characteristics, such as audio signals, where events can occur intermittently within the labeled segment (Turpault et al., 2021). Understanding and mitigating this noise is critical to effectively leverage weak labels in downstream applications (Kumar & Raj, 2016).

The noise in weak labels can be categorized into two types: class label noise (mislabeling event presence or absence in a segment) and segment label noise (mislabeling due to misaligned segment boundaries). While class label noise has been extensively studied (Song et al., 2022; Zhang et al., 2021), the effects of segment label noise remain underexplored. This type of noise significantly affects tasks such as sound event detection (Hershey et al., 2021; Turpault et al., 2021; Shah et al., 2018) and medical image segmentation (Yao et al., 2023). Strategies like pseudo-labeling (Dinkel et al., 2022), robust loss functions (Fonseca et al., 2019), and adaptive pooling operators (McFee et al., 2018) aim to address challenges when training on weak labels. However, fully understanding the impact of weak labels requires quantifying their accuracy (Shah et al., 2018; Turpault et al., 2021).

Current methods typically estimate label noise rates *after* collecting labels (Song et al., 2022), employing techniques like noise transition matrices (Li et al., 2021) or cross-validation (Chen et al., 2019). In contrast, predicting label noise rates *before* data collection remains largely unexplored. This is particularly challenging when the noise stems from human annotators, as it is difficult to formalize. In cases involving partially automated processes, however, the noise introduced by the automated component can often be modeled under specific assumptions.

In this work, we model the automated component of a commonly used weak labeling method for segmentation tasks: fixed-length weak labeling (FIX). We quantify the segment label noise of this process, and study the expected label accuracy. This method, commonly employed in sound event detection, involves annotators providing presence or absence labels for fixed-length segments of the data (automated component), rather than specifying precise event boundaries. By simplifying the labeling process, FIX weak labeling reduces annotation effort but introduces segment label noise when segments misalign with the actual onsets and offsets of events. To benchmark this approach, we compare it to an oracle weak labeling method, ORC weak labeling, which assigns presence or absence labels to segments derived using the true onsets and offsets of the events.

Figure 1 illustrates the trade-offs between annotation cost and label accuracy for the ORC and FIX weak labeling methods. ORC weak labeling achieves perfect label accuracy by aligning the segments with the ground truth presence events (green) using a minimal number of annotated segments. In contrast, FIX weak labeling shows varying accuracy depending on segment length: shorter segments improve alignment ($B = 30$) but require more annotations, while longer segments ($B = 7$) reduce cost at the expense of accuracy. In addition, too short segments ($B = 60$) can lead to the annotator missing event presence. These trade-offs

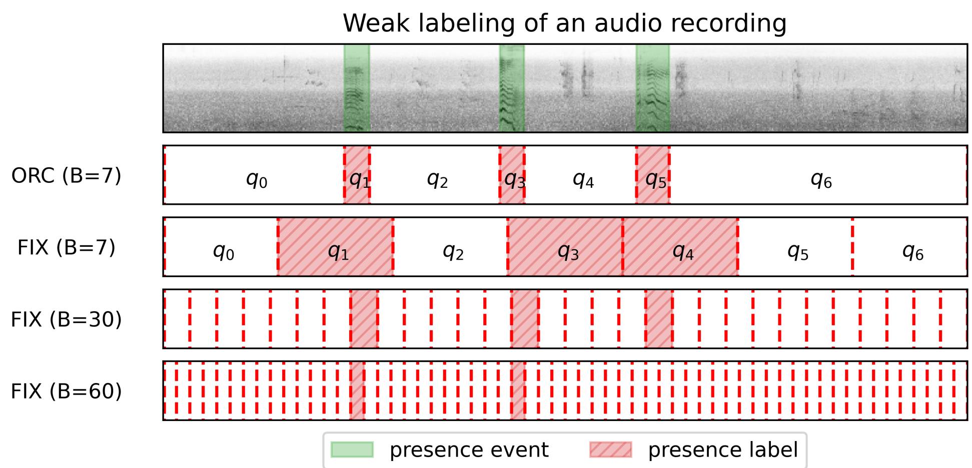

Figure 1: Resulting presence (red) and absence (white) labels from ORC and FIX weak labeling for an audio recording with three presence events (green). ORC weak labeling assigns labels to ground truth segments, achieving perfect label accuracy with $B = 7$ labels. FIX weak labeling, shown for different segment lengths ($B = 7$, $B = 30$, $B = 60$), introduces segment label noise as segments misalign with events. Longer segments reduce annotation cost but increase noise, while shorter segments align better but require more annotations. Note that too short segments ($B = 60$) may lead to the annotator missing the presence of the event because it does not cover a large enough fraction of it.

are central to understanding how FIX weak labeling can be used effectively. By analyzing the FIX weak labeling method, we provide a theoretical framework to guide data collection efforts.

In summary, our contributions include:

- Closed-form expressions for label accuracy and annotation cost in FIX and ORC weak labeling, made tractable by assuming an annotator model and a simplified data distribution.
- A simulation study demonstrating that our theoretical framework generalizes to more complex data distributions and serves as an upper bound for the accuracy of FIX weak labeling.
- A theoretical foundation for developing adaptive weak labeling methods that better approximate ORC weak labeling, such as (Martinsson et al., 2024) for sound event detection and (Kim et al., 2023) for image segmentation.

Our analysis focuses on one-dimensional data, and the assumptions are justified by common characteristics in bioacoustic sound events. These time-localized, non-stationary animal vocalizations often require annotators to hear significant portions of the sound to assign accurate presence labels. Note, however, that while our framework is tailored to this domain, the principles extend to annotation of events in time in other data that shares these characteristics.

2 Problem Setting

The analysis is framed within a multi-pass binary labeling setting. Here, an annotator assigns binary labels (presence or absence) to data segments based on the occurrence of specific sound events. The annotator model abstracts how an annotator interacts with data by labeling segments, without requiring precise knowledge of event boundaries. While inspired by time-localized and non-stationary sound events, this framework is generalizable to any time series with similar characteristics.

It's important to emphasize that, in this weak labeling setting, the concept of overlapping events is not explicitly modeled. Overlapping events from the same class are treated as a single, longer presence event, because presence/absence labels cannot differentiate between individual event instances. For instance, in an audio recording with two birds calling simultaneously, this weak labeling framework simplifies the overlap into a single 'present' event. While this simplification is necessary when studying weak labeling in this setting, it fundamentally restricts our ability to resolve polyphony (the identification of multiple overlapping sound events). We leave the exploration of annotator models capable of providing richer labels to future work; this is beyond the scope of our study.

2.1 The Assumed Data Distribution

A sound event e is defined by its start time $a_e \in \mathbb{R}$, end time $b_e \in \mathbb{R}$, and class $c_e \in \mathcal{C}$, denoted as $e = (a_e, b_e, c_e)$. Audio recordings are assumed to have finite length T, and events are uniformly distributed over the recording.

The uniform distribution reflects the assumption that events are equally likely to appear anywhere in the recording relative to the recording's start time. Consider a person who wants to record an event but does not know when the event will occur. We assume that they are equally likely to start recording at any time before this event.

2.2 The Assumed Annotator Model

For a given sound event class $c \in \mathcal{C}$, the annotator decides the presence or absence of an event e of class c in a data segment $q = (a_q, b_q)$, where $d_q = b_q - a_q$ is the fixed-length of the segment. We will refer to q as a query segment because it is queried for a presence or absence label. Let $l_q \in \{0, 1\}$ denote the weak label indicated by the annotator for query segment q, where $l_q = 1$ indicates presence of an event of class c in q and $l_q = 0$ indicates absence of that event class in q. Detecting the presence of an event requires observing a sufficient fraction of the event within the query segment, formalized as follows:

Definition 1. The *event fraction* is the fraction of the total event duration $d_e = b_e - a_e$ that overlaps with the query segment q,

$$h(e, q) = \frac{|e \cap q|}{d_e}, \tag{1}$$

where $e \cap q$ is the intersection of (a_e, b_e) and (a_q, b_q).

Definition 2. The *presence criterion* $\gamma \in (0, 1]$ is the minimum event fraction required for the annotator to detect the presence of e in q,

$$h(e, q) \geq \gamma. \tag{2}$$

The annotator assigns a presence label ($l_q = 1$) to q if there is sufficient overlap with any presence event e of class c ($h(e, q) \geq \gamma$); otherwise, it assigns an absence label ($l_q = 0$). The parameter γ reflects the annotator's sensitivity: lower γ values indicate sensitivity to smaller event fractions, while higher values require larger fractions.

This framework captures variability in annotator behavior. For example, detecting "human speech" or "bird song" may only require hearing a small fraction of the event (γ closer to 0), while recognizing specific phrases or bird species might demand a near-complete observation (γ closer to 1). The value of γ thus depends on the annotator and the complexity of the event class. This model provides a flexible yet precise way to simulate annotator behavior and quantify their labeling performance. However, it is important to note that this model is deterministic, focusing on temporal alignment between events and the query segment. In practice, human annotation often involves stochastic factors, such as variability in perception and judgment, which are not explicitly modeled here.

2.3 Label Accuracy

Label accuracy measures the alignment between annotator-provided labels and ground truth labels:

Definition 3. The label accuracy is defined as

$$F(e, q, \gamma) = \begin{cases} \frac{|e \cap q|}{d_q}, & \text{if } l_q = 1, \\ \frac{d_q - |e \cap q|}{d_q}, & \text{if } l_q = 0. \end{cases} \quad (3)$$

For instance, consider a 3-second query segment ($d_q = 3$) that overlaps exactly one second ($|e \cap q| = 1$) with a 2-second sound event ($d_e = 2$) of the class bird song ($c =$ "bird song"). The annotator assigns a presence label ($l_q = 1$) with label accuracy $\frac{|e \cap q|}{d_q} = \frac{1}{3}$ if half or less of the event needs to be in the query segment ($\gamma \leq 0.5$). Contrary, the annotator assigns an absence label ($l_q = 0$) with label accuracy $\frac{d_q - |e \cap q|}{d_q} = \frac{3-1}{3} = \frac{2}{3}$ if more than half of the event ($\gamma > 0.5$) needs to be in the query segment. This formulation isolates the segment label noise ($1 - F(e, q, \gamma)$) introduced by the automated component (fixed-length segments) of the FIX weak labeling method.

3 The Label Accuracy and Cost of ORC Weak Labeling

Let us start with the ORC weak labeling method. This method uses a priori information about the event start and end times and is therefore not available in practice, but should be seen as an upper bound on what can be achieved with weak labeling. The start and end times of the true presence and absence events are used to construct the query segments:

$$\mathbb{Q}_{\text{ORC}} = \{(a_0, b_0), (a_1, b_1), \ldots, (a_{B_{\text{ORC}}-1}, b_{B_{\text{ORC}}-1})\} = \{q_0, \ldots, q_{B_{\text{ORC}}-1}\}, \quad (4)$$

where (a_i, b_i) is the ith ground truth presence or absence event. The annotator indicates presence or absence for each of these segments, which by construction results in the ground truth annotations, illustrated in Figure 2. In the example, there are three target events (green), and four absence events, which means that $B_{\text{ORC}} = 7$. In general $B_{\text{ORC}} \in \{2M - 1, 2M + 1\}$, where M denotes the number of presence events. The number of absence events can be fewer than $2M + 1$ if the recording starts or ends with a presence event, however, for simplicity and without losing generality, we will consider $B_{\text{ORC}} = 2M + 1$ as the minimum number of query segments needed for ORC to derive the ground truth. From an annotation cost perspective, this is the most cautious choice, and it is also the most likely outcome. The query accuracy is 1 for each query segment since by construction the fraction of correctly labeled data in each query segment will be 1 when given the correct presence or absence labels.

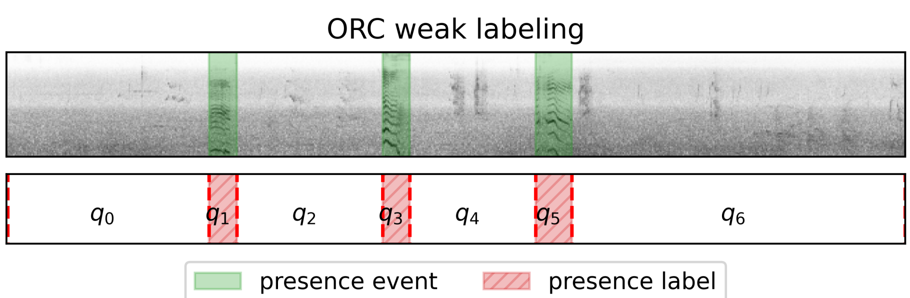

Figure 2: ORC weak labeling of an audio recording with three target events ($M = 3$) shown in green and four absence events. The $B_{\text{ORC}} = 7$, query segments $q_0, \ldots, q_6$ are derived from the ground truth segmentation of the data, and therefore the label accuracy will by definition be 1.

In summary, the ORC weak labeling method produces annotations with label accuracy 1, using the minimum number of query segments needed to achieve this. We use this as a reference on what can be achieved for weak labeling data.

4 The Label Accuracy and Cost of FIX Weak Labeling

The outline of this section is as follows. In Section 4.1 we define the FIX labeling method. In Section 4.2 we derive a closed-form expression for the expected label accuracy of a query segment given that it overlaps with a single event of deterministic event length. We note that it is only in the cases of overlap between a query segment and an event that a presence label can occur under the assumed annotator model, and that the expectation in label accuracy over these cases therefore can be viewed as the expected presence label accuracy. For the remainder of the paper we will simply write expected label accuracy when referring to the expectation over the overlapping cases, unless explicitly stated otherwise.

In the same section we derive the optimal query length with respect to the expected label accuracy, the maximum expected label accuracy and the number of query segments needed (proxy for annotation cost). In Section 4.3 we explain how the expression for expected label accuracy can be used in the case of a single event of stochastic length, and in Section 4.4 we explain under which conditions this can be used when multiple events can occur. Finally, we derive a closed form expression for the expected label accuracy of an audio recording with multiple events of stochastic length in Section 4.5, and provide an alternative interpretation of the theory in Section 4.6.

4.1 The FIX Weak Labeling Method

The FIX weak labeling method, commonly used in practice, splits the audio recording into fixed and equal length query segments, and then an annotator is asked to provide either a presence or absence label for each of the query segments. Let B_{FIX} denote the number of query segments used, then the query segments for an audio recording of length T are defined as

$$\mathbb{Q}_{\text{FIX}} = \{(a_0, b_0), (a_1, b_1), \ldots, (a_{B_{\text{FIX}}-1}, b_{B_{\text{FIX}}-1})\} = \{q_0, \ldots, q_{B_{\text{FIX}}-1}\}, \tag{5}$$

where the start and end timings of each query segment is $q_i = (a_i, b_i) = (id_q, (i+1)d_q)$ and the fixed query segment length is $d_q = T/B_{\text{FIX}}$. We illustrate this in Figure 3, where the presence criterion for the annotator is $\gamma = 0.5$. There are three presence events and four absence events, and using only $B_{\text{FIX}} = 7$ query segments results in annotations with an average label accuracy that is lower than 1.

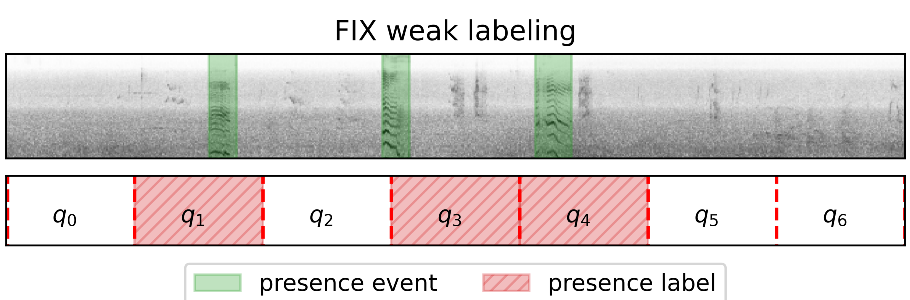

Figure 3: Illustration of the FIX weak labeling method. The audio recording contains presence events (green). The FIX method divides the recording into fixed-length query segments (e.g., q_0 to q_6). Note how the alignment between segments and presence events affects the accuracy of presence labels (red hatched).

We want to find an expression for the expected label accuracy for a given data distribution and query segment length. In addition, we want to understand the query length that maximize the expected label accuracy.

4.2 The Expected Label Accuracy of a Query Segment given Event Overlap

To derive a tractable closed-form expression we analyze a simplified data distribution, consisting of audio recordings of length T that always contain a single event of deterministic length d_e. This is arguably the simplest data distribution to annotate, and the results can therefore be viewed as an upper bound on the expected label accuracy for any more complex data distribution.

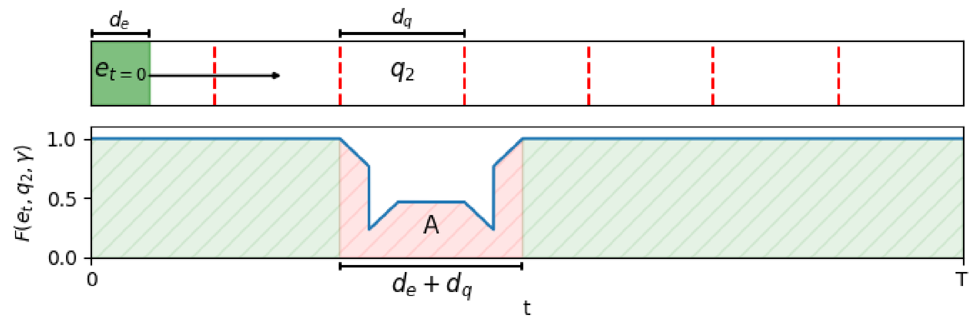

Figure 4: *Top panel:* A single event (e_t) of length d_e can occur at various end times (t) within the recording of length T. *Bottom panel:* The resulting label accuracy for query segment q_2 (arbitrarily chosen for illustration) of length d_q as a function of the event's end time (t). Overlap between the event and the query segment leads to segment label noise and a reduced label accuracy, which in this case occur when $t \in [a_2, a_2 + d_e + d_q]$ where a_2 is the start time of q_2. The red hatched area (A) represents the cumulative label accuracy during these overlapping scenarios.

The setup is illustrated in the upper panel of Figure 4, where a single event e_t of length d_e can occur at any time $t \in [0, T]$ (indicated by the arrow). The bottom panel of Figure 4 shows the label accuracy for a specific query segment (q_2) as the end time (t) of the event varies. The area (A) highlighted in hatched red indicates the label accuracy in the cases of overlap between the query segment and the event, and the area in hatched green indicate the label accuracy in the cases of no overlap, which is by the definition of the annotator model is always 1. Crucially, while this figure illustrates the accuracy for query segment q_2, the shape of this accuracy function remains the same for other query segments; only its position along the x-axis would change.

To simplify the mathematical analysis, without loss of generality, we can fix the query segment to start at time 0, $q = (0, d_q)$, and represent the event with its ending time t as $e_t = (t - d_e, t)$. In this way, $t \in [0, d_e + d_q]$ describes all possible overlap occurrences. That is, when $t = 0$ the event ends at the start of the query segment, and when $t = d_e + d_q$ the event starts at the end of the query segment. To formalize this, we can express the expected label accuracy in case of overlap by integrating over all possible event end times (t) where overlap occurs:

$$\mathbb{E}_{\mathrm{t}\sim p}\left[F(e_{\mathrm{t}}, q, \gamma)\right] = \int_0^{d_e+d_q} F(e_t, q, \gamma) p(t) \mathrm{d}t, \tag{6}$$

$$= \frac{1}{d_e + d_q} \int_0^{d_e+d_q} F(e_t, q, \gamma) \mathrm{d}t \tag{7}$$

$$= \frac{A}{d_e + d_q}. \tag{8}$$

where $\mathrm{t} \sim p$ denotes a random variable t distributed according to a distribution p, and $p(t)$ denotes the probability of realization t. Since we assume that the sound event can occur anywhere in the audio recording with equal probability we get $p(\mathrm{t}) = 1/(d_e + d_q)$, and by observing that the integral $\int_0^{d_e+d_q} F(e_t, q, \gamma)\mathrm{d}t$ describes the hatched red area denoted A in Figure 4 we arrive at the final expression in Eq. 8.

Remember that absence labels can occur when there is no overlap (always correct) and when there is overlap but the presence criterion is not fulfilled, and presence labels can only occur when there is overlap and the presence criterion is fulfilled. Therefore, inaccurate labels only occur in the case of overlap. The expected label accuracy in the case of overlap therefore describes the accuracy of the labels when segment label noise can occur, which happens around the boundaries of the true event.

In Appendix A.1 we show how to express A in terms of the event length d_e, the query segment length d_q and the presence criterion γ under the assumption that the annotator presence criterion can be fulfilled ($d_q \geq \gamma d_e$), and that it can not be fulfilled ($d_q < \gamma d_e$). Finally, we arrive at the following four main theorems:

Theorem 1. The expected label accuracy in case of overlap between a query segment q of length d_q and a single event e of deterministic length d_e is

$$f(d_q) = \mathbb{E}_{t\sim p}\left[F(e_t, q, \gamma)\right] = \begin{cases} \frac{d_e\left(2\gamma d_q - 2\gamma^2 d_e + d_q\right)}{d_q(d_e+d_q)}, & \text{if } d_q \geq \gamma d_e, \\ \frac{d_q}{d_e+d_q}, & \text{if } d_q < \gamma d_e, \end{cases} \tag{9}$$

when the presence criterion for the annotator is γ.

Proof. See Appendix A.1 for the proof. We show how to express the area A in Eq. 8 in terms of d_e, d_q and γ for the two assumptions: $d_q \geq \gamma d_e$, and $d_q < \gamma d_e$. □

Theorem 2. The query length that maximizes the expected label accuracy in case of overlap for a given event length d_e is

$$d_q^* = d_e \gamma \frac{2\gamma + \sqrt{4\gamma^2 + 4\gamma + 2}}{2\gamma + 1}. \tag{10}$$

Proof. See Appendix A.2 for the proof. We compute the derivative of $f(d_q)$ with respect to d_q, and show that d_q^* is the maximum. □

Theorem 3. The maximum expected label accuracy in case of overlap between a query segment of length d_q and an event of length d_e when $d_q \geq \gamma d_e$ is

$$f^*(\gamma) = f(d_q^*) = 2\gamma\left(2\gamma + 1 - \sqrt{4\gamma^2 + 4\gamma + 2}\right) + 1 \tag{11}$$

Proof. See Appendix A.3 for the proof. We substitute d_q for d_q^* in Eq. 9. □

Theorem 4. The number of queries B^*_{FIX} (cost) that are needed by FIX to maximize the expected label accuracy in case of overlap for an audio recording of length T when $d_e = 1$ is

$$B^*_{\text{FIX}} = \frac{T}{d_q^*}. \tag{12}$$

Proof. $T/B^*_{\text{FIX}} = d_q^*$, which by Theorem 2 leads to maximum label accuracy. □

In summary, Theorem 1 gives us an expression $f(d_q)$ for the expected label accuracy when query segments of length d_q are used to detect events of length d_e and the presence criterion for the annotator is γ. We use this to find the query segment length d_q^* that maximize the expected label accuracy, leading to Theorem 2. Theorem 2 show the query segment length d_q^* that maximizes expected label accuracy for a given event length and annotator criterion. Further, by inserting d_q^* into Theorem 1, $f^*(\gamma) = f(d_q^*)$, we get Theorem 3, which is the maximum achievable expected label accuracy for a given annotator criterion γ. We have omitted the case $d_q < \gamma d_e$ when deriving $f^*(\gamma)$, since maximizing the expected label accuracy in the case when the annotator presence criterion can not be fulfilled is not very interesting, since we can not get presence labels. Note that $f^*(\gamma)$ is a function of only γ, meaning that the maximum expected label accuracy is independent of the target event length when considering a single deterministic event. Finally, Theorem 4 show that an annotator needs to weakly label B^*_{FIX} query segments for each audio recording to achieve the maximum label accuracy in expectation, which can be seen as a proxy for annotation cost.

There is arguably no simpler audio data distribution to annotate than when recordings only contain a single event of deterministic length (except for when no event occurs at all). We can therefore treat $f^*(\gamma)$ as an upper bound on the maximum expected label accuracy for any audio distribution. We demonstrate this empirically in the results in Section 6. However, in practice audio recordings often contain events that vary both in length and number. Let us therefore consider how the derived theory can be useful also in these cases.

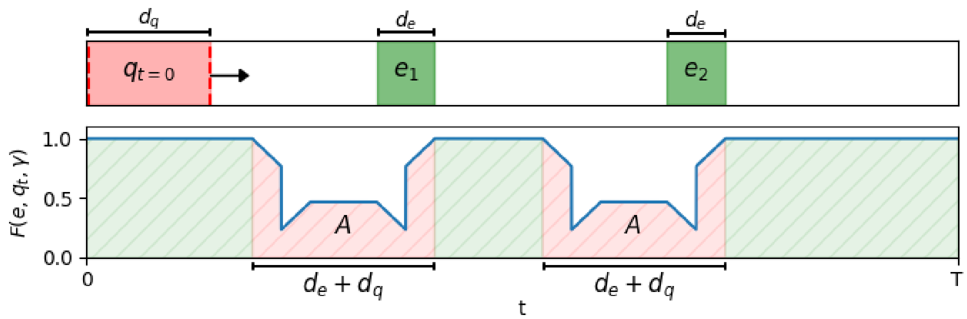

Figure 5: *Top panel:* Two events ($M = 2$) of length d_e that are fixed in time within a recording of length T, and a query segment $q_t = (-d_q + t, t)$. *Bottom panel:* The resulting label accuracy of q_t for $t \in [0, T - d_q]$, simulating that the q_t can appear anywhere at random in time in relation to the events. As before, when there is overlap between the query segment and an event the label accuracy is below 1, otherwise it is always 1.

4.3 Stochastic Event Length

Events may vary in length according to some event length distribution. Let $p(d_e)$ denote the probability of the outcome that an event has length d_e, and let $d_e \sim p(d_e)$ denote that d_e is a sample from that distribution. The expected label accuracy over a distribution of event lengths for a given γ and query segment length d_q can then be computed as

$$\mathbb{E}_{d_e \sim p(d_e)} [f(d_q)] = \int_0^\infty f(d_q) p(d_e) \mathrm{d}d_e. \tag{13}$$

While we do not provide a closed form solution for this, we can solve the integral in Eq. 13 by numerical integration. Note that d_q^* in Theorem 2 depends on the single event length d_e, and to find it for a distribution we would need to solve Eq. 13 for a range of d_q and find the one that leads to the best label accuracy. However, for some event length distributions, setting d_e to the average of the distribution turns out to be a good heuristic. We perform a simulation study in Section 6.1.2 to support these claims.

4.4 Multiple Events

There may be multiple (M) events present in a given audio recording. In Figure 5 we show the label accuracy for all possible occurrences of a query segment q_t in a recording with two events ($M = 2$). Note that we have put the subscript t on the query segment (q_t) instead of the event as in the prior analysis. This formulation is entirely equivalent, but when talking about multiple events it is more intuitive to consider them as fixed in time for a given recording, and that the query segments occur relative them at random. There are now two regions where overlap occurs, one around e_1 and one around e_2. On average we get $2A/2(d_e + d_q) = A/(d_e + d_q) = f(d_q)$. That is, the theory we derived for the single event case explains the multiple event case.

However, for this to hold we need to assume that for any event the closest other event is least d_q away in time. In Figure 5 this holds since the start of e_2 is at least d_q away from the end of e_1. If this assumption holds then the expected label accuracy for multiple events is $f(d_q)$. The assumption is plausible if events are sparse in relation to d_q. Note that $d_q^* \in (0, d_e \frac{2+\sqrt{10}}{3}]$ for $\gamma \in (0, 1]$ according to Theorem 2. That is, when considering the optimal query length d_q^* this assumption translates to that events should be no closer than approximately $1.72 d_e$ for $\gamma = 1$, $0.81 d_e$ for $\gamma = 0.5$, and 0 for $\gamma \to 0$. We perform a simulation study in Section 6.1.3 to see the effect of breaking this assumption, and we leave it to future work to derive the expected label accuracy in case of overlap for multiple events.

4.5 The Expected Label Accuracy of an Audio Recording

We now know the expected label accuracy of a query segment given event overlap, and how to use this for a stochastic event lengths and multiple events. We can use this to derive an expression for the expected label accuracy of and audio recording of finite length (T) that has multiple (M) stochastic event lengths ($d_e \sim p(d_e)$).

Theorem 5. The expected label accuracy for an audio recording of length T, with M events of stochastic event length $d_e \sim p(d_e)$ that are spaced at least d_q apart is

$$\mathbb{E}_{d_e \sim p(d_e)} \left[-\frac{2Md_e^2\gamma^2}{Td_q} + \frac{2Md_e\gamma}{T} - \frac{Md_q}{T} + 1 \right]. \tag{14}$$

Proof. We will do this proof by picture. In Figure 5 we have two events ($M = 2$), in general for M events the accumulated label accuracy in the cases of overlap is MA (the sum of the hatched red areas), the total amount of overlapping cases is $M(d_e + d_q)$ and the total amount of non-overlapping cases is therefore $T - M(d_e + d_q)$ for an audio recording of length T. In the case of no overlap, the label accuracy is always 1, which means that the accumulated label accuracy in the case of no overlap (sum of the green hatched areas) is $T - M(d_e + d_q)$. Normalizing for the entire duration of the recording we arrive at

$$\frac{AM + T - M(d_e + d_q)}{T} = -\frac{2Md_e^2\gamma^2}{Td_q} + \frac{2Md_e\gamma}{T} - \frac{Md_q}{T} + 1, \tag{15}$$

and as before we can simply compute an expectation over the event length distribution. □

Theorem 5 tells us the expected label accuracy under FIX weak labeling with query segment length d_q for an audio recording of length T, with M events of stochastic event length $d_e \sim p(d_e)$. If we want to account for class label noise, where the annotator gives the wrong label with probability ρ, this can be included by simply scaling the whole expression in Eq. 14 by $(1 - \rho)$. That is, the expected label accuracy for the cases of overlap allows us to express a variety of things about the expected label accuracy of an audio recording.

However, note that we have T in the denominator of all terms except the term that is 1, meaning that if we let T approach ∞, then the expected label accuracy approaches 1. That is, considering the accuracy of both absence and presence labels equally can lead to hiding the effect that we want to understand in this paper, which is the effect of d_q on the accuracy of the presence labels. We could derive a balanced accuracy in a similar way as above, but instead we choose to continue our analysis looking only at the expected label accuracy in the case of overlap.

4.6 Expected Label Accuracy given Overlap when $d_q = \delta d_e$

As a result of the proof for Theorem 3 in Appendix A.3 we get an alternative dimensionless interpretation of the expected label accuracy when the query segment length is expressed as a factor of the event $d_q = \delta d_e$,

$$f(\delta d_e) = \frac{(2\gamma + 1)\delta - 2\gamma^2}{\gamma(1 + \gamma)}, \tag{16}$$

and an expression for the ratio that maximizes it

$$\delta^* = \frac{d_q^*}{d_e} = \gamma \frac{2\gamma + \sqrt{2\gamma^2 + 2\gamma + 1}}{2\gamma + 1}. \tag{17}$$

This alternative formulation illustrates that it is the ratio $\delta = d_q/d_e$ that affects the expected label accuracy of a single event, and not the absolute lengths d_q and d_e. Further, we can use this interpretation to rewrite Theorem 5 as

$$\mathbb{E}_{\delta \sim p(\delta)} \left[\frac{Md\delta\,(-\delta + 2\gamma) - 2Md\gamma^2 + T\delta}{T\delta} \right], \tag{18}$$

where δ denotes a random variable with probability distribution $p(\delta)$.

5 Simulating the Label Accuracy of FIX Weak Labeling

To validate the theory, we simulated FIX labeling of various audio recording distributions and compared the average simulated label quality with the theoretical results from Section 4.2. The code used for these simulations is provided in the supplementary material.

We generated 1000 audio recordings of length $T = 100$ seconds for each configuration. The number of events, M, and the event length distributions varied across simulations, as detailed below:

- **Single Event with Deterministic Length:** We simulated recordings with $M = 1$ event of deterministic length $d_e = 1$ second.
- **Single Event with Stochastic Length from Normal Distributions:** We drew event lengths from two normal distributions with the same mean but different variances ($\mathcal{N}(3, 0.1)$ and $\mathcal{N}(3, 1)$), and from two normal distributions with different means but the same variance ($\mathcal{N}(0.5, 0.1)$ and $\mathcal{N}(5, 0.1)$). For these simulations, $M = 1$.
- **Single Event with Stochastic Length from Gamma Distributions:** We sample event lengths from two gamma distributions (offset by 0.5 seconds due to computation cost) with different shape parameters but the same scale parameter ($\text{Gamma}(0.8, 1) + 0.5$ and $\text{Gamma}(0.2, 1) + 0.5$) with $M = 1$.
- **Single Event with Stochastic Length from Real Length Sample:** We used the event length distributions for dog barks and baby cries from the NIGENS dataset (Trowitzsch et al., 2019) with $M = 1$.
- **Multiple Events with Deterministic Length:** We simulated recordings with multiple events ($M = 30$ and $M = 50$) where each event had a deterministic length of $d_e = 1$ second.

For recordings with stochastic event lengths or multiple events, the length of each of the M events was sampled from the specified distribution. Each sampled event was then placed randomly within the recording. The start time a_e of each event was drawn uniformly at random from $[0, T - d_e]$. If multiple events were present, overlapping events were merged into one presence event. For each generated audio recording, we simulated FIX labeling using different annotator presence criteria $\gamma \in [0.01, 0.99]$ and a range of query segment lengths d_q. The query segment lengths were linearly spaced between a small fraction of the minimum event length observed in the distribution and a value several times the maximum observed event length.

We then computed the average label accuracy over the query segments that overlaps with an event in each recording. For each query segment q we check if the annotator presence criterion ($h(e, q) \geq \gamma$) is fulfilled for any event $e \in E$, where E is the set of all events that overlap with q. If this is true for any of the events then q is given a presence label ($l_q = 1$) otherwise it is given an absence label ($l_q = 0$). The label accuracy is then computed in a similar way as in Eq. 3, but since we can now have multiple events overlapping with the same query segment, we need to consider the union of all overlapping events $\cup_{e \in E} e$ when computing the label accuracy of assigning label l_q to that query segment. The total amount of overlap becomes $|(\cup_{e \in E} e) \cap q|$ instead of $|e \cap q|$. However, when $M = 1$ this is equivalent to Eq. 3 ($|(\cup_{e \in E} e) \cap q| = |e \cap q|$), since $|E| = 1$.

In this way, we simulated the effect of breaking the assumption that events are spaced at least d_q apart, and could better understand the effect this had when compared to the derived theory. Finally, for each considered γ, we empirically determined the maximum average label accuracy across all tested query lengths and the corresponding optimal query length. These empirical results were then compared to the theoretical predictions.

6 Results

In this section we present the results of the simulated annotation process, and show how these connect to the derived theory. We start by looking at the expected label accuracy and the query segment length that maximize the expected label accuracy for FIX and ORC weak labeling, and then we relate this to the annotation cost.

6.1 Expected Label Accuracy given Overlap

We evaluate how different annotator presence criteria (γ) influence the achievable label accuracy given overlap under FIX weak labeling. We first examine the case of a single event with a deterministic length, then extend our simulation study to stochastic event lengths, and finally to multiple events occurring within the same recording.

6.1.1 Single Event with Deterministic Length

The simulated results are derived using the simulation setup described in section 5, with $M = 1$ (a single event) and $d_e = 1$ (deterministic length). In Figure 6, we show the maximum expected label accuracy given overlap (left) and the corresponding query length that maximize the label accuracy (right) for different γ. $f^*(\gamma)$ is the maximum expected label accuracy achievable with annotator presence criterion γ for the considered event length. We can see that the simulated average label accuracy closely follows the expected label accuracy, and that the corresponding segment length leading to this maximum is the same in theory and simulation.

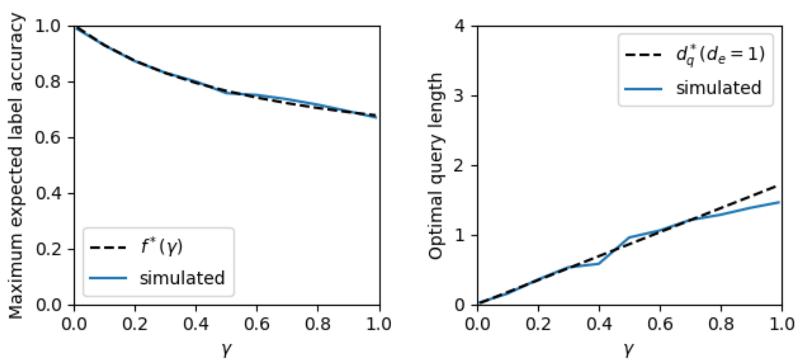

Figure 6: In the left panel we show the maximum expected label accuracy, $f^*(\gamma)$, for different γ, and the average maximum label accuracy from the simulations. In the right panel we show the query length that leads to this maximum label accuracy in theory, for $d_e = 1$, and in simulation. The theory follows the simulations well.

In Figure 6 we see that if the annotator needs to hear more than 50% of the sound event to detect presence ($\gamma = 0.5$) then the highest achievable label accuracy is $f^*(0.5) \approx 0.76$. This means that on average there is around 34% segment label noise around the presence labels. We also see that the query length that gives the maximum label accuracy is $d_q^* \approx 0.81$. The gap to the ORC weak labeling method which always gives a label accuracy of 1, is large especially for large γ. In general, we can see how the maximum label accuracy deteriorates with a growing γ, and which query segment length to choose to maximize label accuracy in expectation.

6.1.2 Single Event with Stochastic Length

We now consider stochastic event lengths. We do this to better understand the effect of the event length distribution on the maximum expected label accuracy and the optimal query length. We solve the integral in Eq. 13 by numerical integration over different event length distributions, and compare with the theory derived for a single deterministic event length and simulations. In each figure we present the derived theoretical rules $f^*(\gamma)$ and d_q^* for the simplified event length distribution, the results from integration of Eq. 13 with different event length distributions $p(d_e)$ (numerical), and the simulated results using the procedure described in

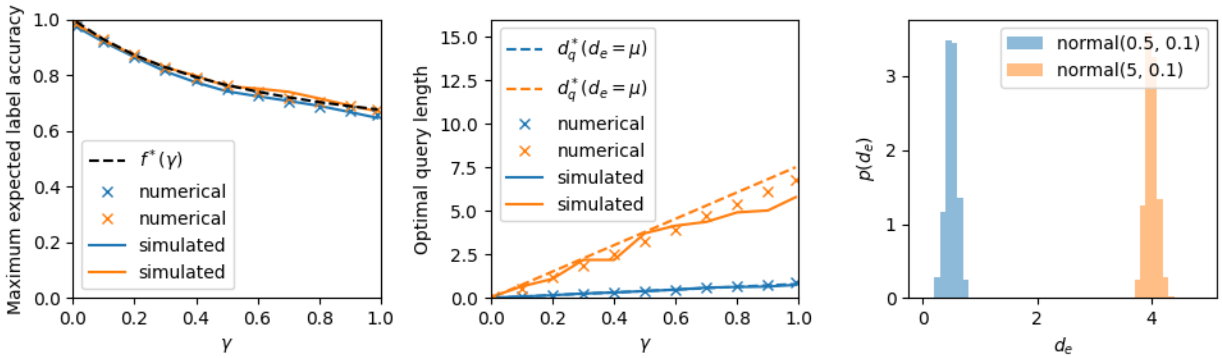

Figure 7: We validate the theory for stochastic event lengths drawn from two normal distributions with different means, but the same variance. We show the expected label accuracy (left panel), the optimal query length (middle panel), and the considered event length distributions (right panel).

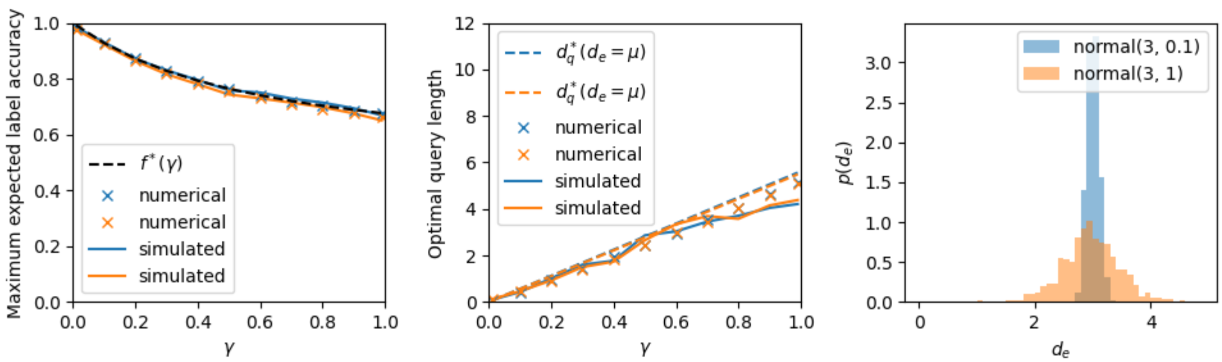

Figure 8: We validate the theory for stochastic event lengths drawn from two normal distributions with different variance, but the same mean. We show the expected label accuracy (left panel), the optimal query length (middle panel), and the considered event length distributions (right panel).

section 5 (simulated) where event lengths are sampled from different distributions. Note that, since d_q^* is derived for a deterministic event length d_e, and require a choice of this value, we set d_e to the average event length (μ) for each distribution in these experiments as a heuristic. We then present the maximum expected label accuracy for different γ (left in figures) and the query segment length that maximizes the expected label accuracy (middle in figures), and the histogram for the considered event length distributions (right in figures).

In Figure 7 and Figure 8 we see that the mean and variance of the normal distribution have a small (if any) effect on the maximum expected label accuracy, but the mean does affect which query segment length that maximizes the expected label accuracy. We also see that d_q^* follows the simulated and numerical optimal query length well for all considered normal distributions, when d_e is set to the average event length (μ) for the considered event length distribution. The average event length can be used as a heuristic value if we only know the average and not the true distribution to integrate over.

In Figure 9 we can see that a gamma distribution does affect the maximum expected label accuracy, and that simply setting d_e to the average event length of the distribution leads to underestimating the optimal query length. Since it is not possible to optimize for both short and long events at the same time using FIX weak labeling, this type of distribution is quite challenging.

In Figure 10 we validate the theory against a real sample of event lengths from either baby cries or dog barks. Numerical integration between the derived expression and the histogram predicts the simulations well.

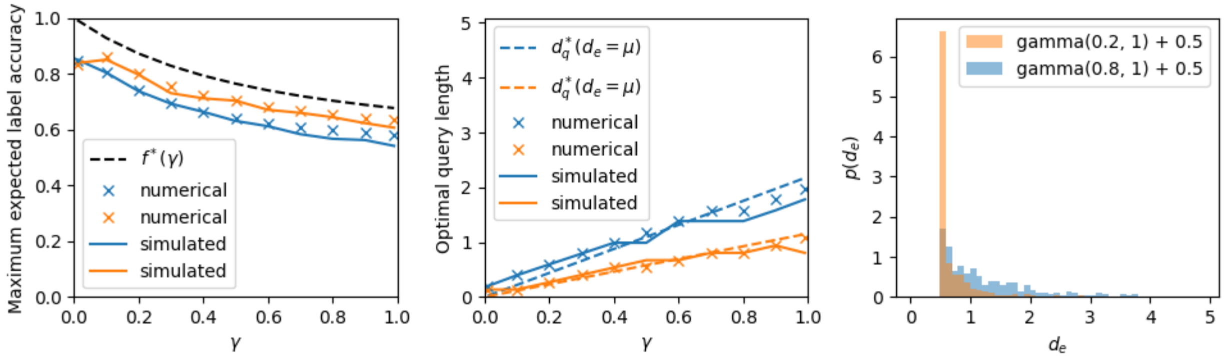

Figure 9: We validate the theory for stochastic event lengths drawn from two gamma distributions with different shape parameters, but the same scale parameter. We show the expected label accuracy (left panel), the optimal query length (middle panel), and the considered event length distributions (right panel).

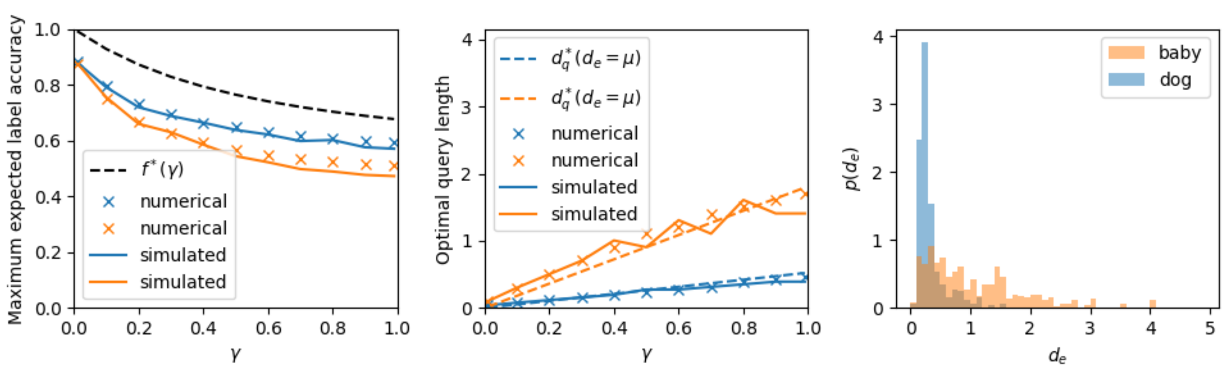

Figure 10: Barking dog and crying baby event length distributions from the NIGENS dataset (Trowitzsch et al., 2019). These annotations have been made with a strong guarantee for high quality onsets and offsets.

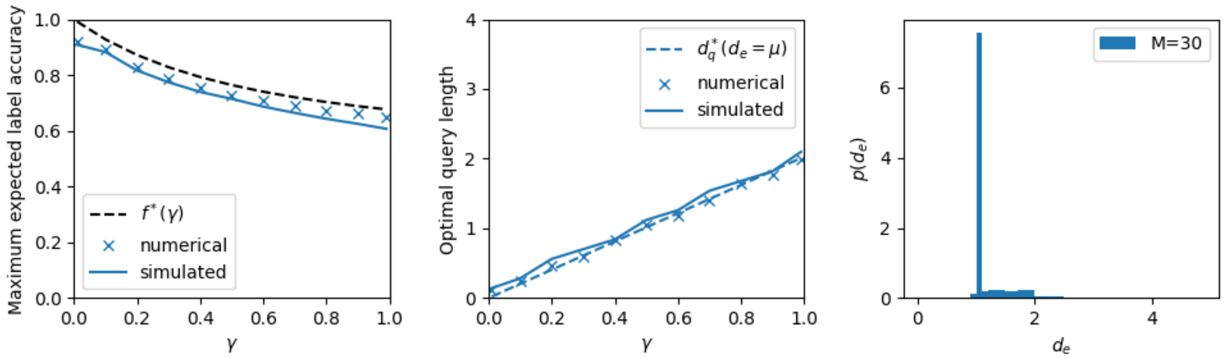

Figure 11: We validate the theory for multiple events of length $d_e = 1$. We show the expected label accuracy (left panel), the optimal query length (middle panel), and the considered event length distributions (right panel). Note that presence events longer than 1 can occur if two or more events overlap. We sample 30 events with event length $d_e = 1$ occur at random for each audio recording in this simulation.

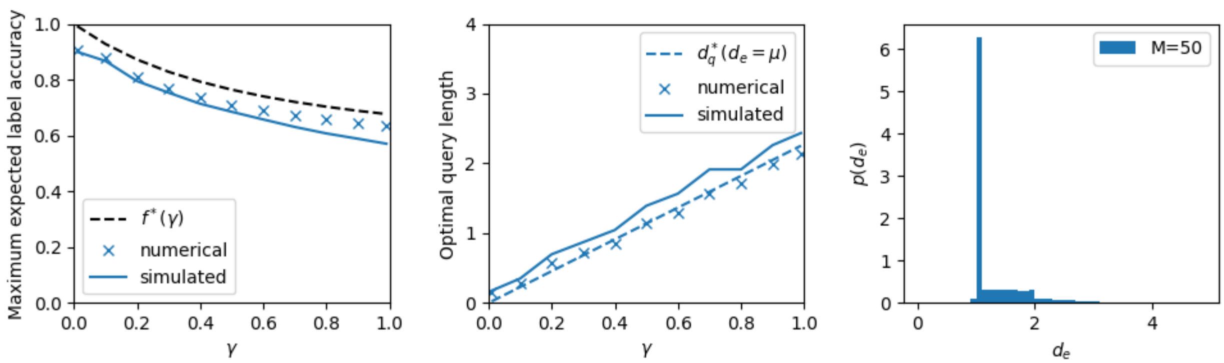

Figure 12: We validate the theory for multiple events of length $d_e = 1$. We show the expected label accuracy (left panel), the optimal query length (middle panel), and the considered event length distributions (right panel). Note that presence events longer than 1 can occur if two or more events overlap. We sample 50 events with event length $d_e = 1$ occur at random for each audio recording in this simulation.

6.1.3 Multiple Events with Stochastic Length

In these simulations we allow multiple events to occur in the same recording ($M > 1$). In Figure 11 we show the results of sampling 30 events of length $d_e = 1$ for each audio recording. This does have a an effect on the expected maximum label accuracy and the corresponding query length, but not (that) large. In Figure 12 we show the results of sampling 50 events of length $d_e = 1$ for each audio recording. This is an extreme case, where the event density of the recording is very high.

6.2 Annotation Cost for Maximum Expected Label Accuracy given Overlap

Achieving maximum expected label accuracy comes at a cost, and understanding this cost trade-off is essential for practical annotation efforts. The cost model we employ accounts for both the time spent listening to audio and the effort required to label presence or absence events.

6.2.1 Formalizing the Cost Model

The derived theory for the optimal query length allows us to analyze the cost of achieving maximum expected label accuracy under different annotator models for FIX weak labeling. We assume that the whole audio

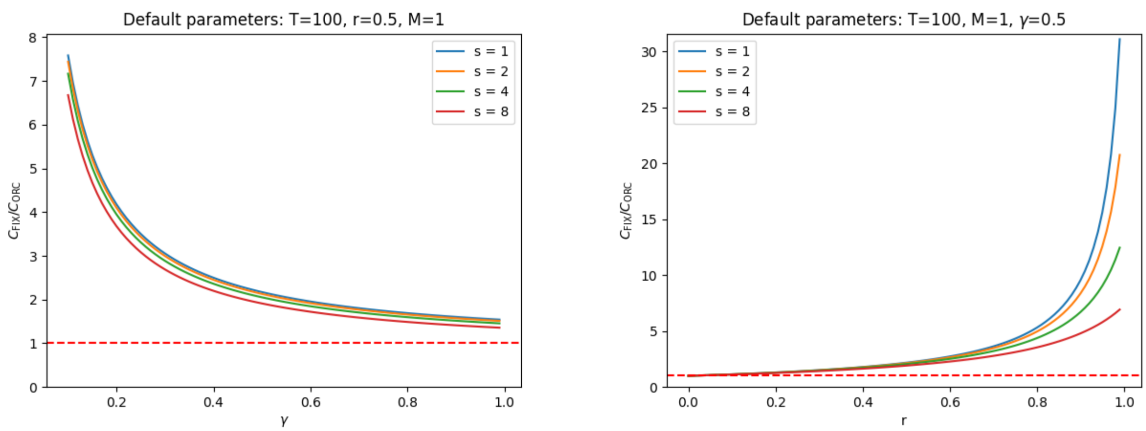

Figure 13: The relative cost of FIX and ORC for varying annotator criteria γ (left), and cost ratios r (right). The default parameters are: $T = 100$, $r = 0.5$, $M = 1$ and $\gamma = 0.5$. We simulate overestimating the number of needed queries $B_{\text{ORC}} = s(2M + 1)$ by a factor of s for $s \in \{1, 2, 4, 8\}$ to see how this affects the relative cost. The cost of FIX is greater than the cost of ORC above the dashed red line where the cost ratio is 1.

recording of length T is listened to. The key difference in cost between the FIX and ORC weak labeling method is the number of segments (B) that need to be given a presence or absence label. We formalize a cost model as:

$$C(T, B) = (1 - r)T + rB, \tag{19}$$

where $1 - r$ represents the cost of listening to one second of audio (cost per second), and r represents the cost of answering a query (cost per query). The term $(1 - r)T$ therefore represents the cost of listening to T seconds of audio, and the term rB the cost of assigning B presence or absence labels. Using this cost model, we calculate the cost of annotating an audio recording of length T with M sound events of length $d_e = 1$ using either FIX or ORC weak labeling. For FIX, the number of queries that maximize expected label accuracy is given by $B^*_{\text{FIX}} = T/d^*_q$ (see Theorem 4). For ORC, achieving an expected label accuracy of 1 requires at least $B^*_{\text{ORC}} = 2M + 1$ queries.

In practice, we do not know the number of events M. To explore potential overestimation of M when, for example, using a weak labeling process that tries to mimic ORC weak labeling, we model B_{ORC} as a multiple of the necessary number of queries: $B_{\text{ORC}} = sB^*_{\text{ORC}}$, where $s \in \{1, 2, 4, 8\}$ represents the degree of overestimation. This approach captures scenarios where the number of events are either precisely estimated ($s = 1$) or significantly overestimated ($s = 8$) during the annotation process. In practice, B_{ORC} could be set based on a bound on M. For example, by estimating a maximum expected number of sound events in a recording, $M_{\max}$, based on knowledge of typical event density, or characteristics of the audio recording. We assume that overestimation by more than a factor of 8 is unlikely. The relative cost between FIX and ORC weak labeling can then be computed as:

$$\frac{C_{\text{FIX}}}{C_{\text{ORC}}} = \frac{C(T, B^*_{\text{FIX}})}{C(T, B_{\text{ORC}})}, \tag{20}$$

where a ratio larger than 1 indicates that FIX is more costly than ORC, and a ratio smaller than 1 indicates that FIX is less costly than ORC.

6.2.2 Effect of annotator criteria (γ) and cost ratio (r).

Figure 13 (left) shows the relative cost for varying annotator criteria $\gamma \in [0.1, 1]$. As $\gamma \to 0.1$, the cost of FIX increases sharply, reflecting the need for an infinitely large number of queries to achieve an expected label accuracy of 1. In practice, achieving perfect accuracy with FIX is infeasible due to the associated cost. For higher γ, the cost of FIX becomes more comparable to ORC. However, combining this with Theorem 3

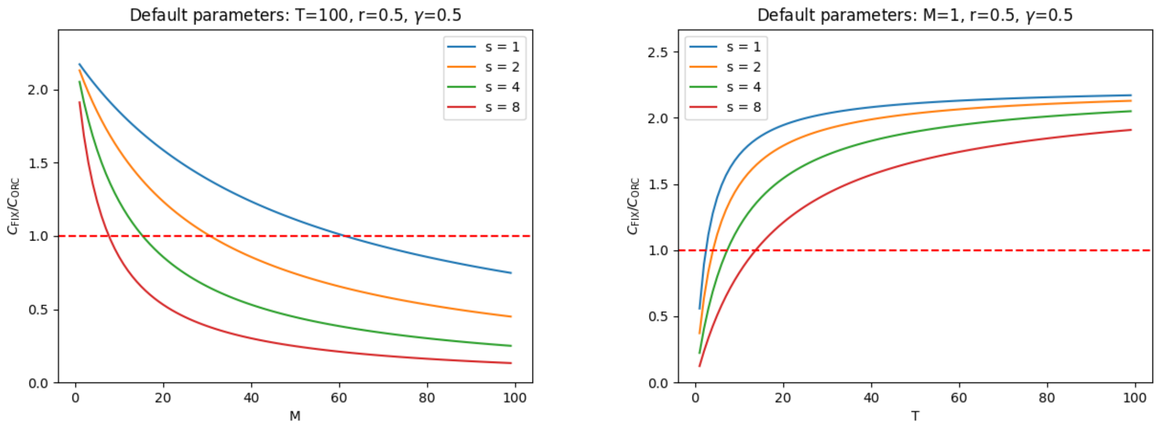

Figure 14: The relative cost of FIX and ORC for varying number of sound events M (left) and recording lengths T (right). The default parameters are: $T = 100$, $r = 0.5$, $M = 1$ and $\gamma = 0.5$. We simulate overestimating the number of needed queries $B_{\text{ORC}} = s(2M + 1)$ by a factor of s for $s \in \{1, 2, 4, 8\}$ to see how this affects the relative cost. The cost of FIX is greater than the cost of ORC above the dashed red line where the cost ratio is 1.

reveals that FIX can either match ORC in cost but with lower expected accuracy or achieve similar accuracy at a much higher cost.

The right panel of Figure 13 examines the impact of the cost ratio r. Across all tested values, ORC remains less costly than FIX in the default setting ($T = 100$, $r = 0.5$, $\gamma = 0.5$, $M = 1$). This confirms that the relative cost advantage of ORC is robust to changes in r.

6.2.3 Effect of number of events (M) and recording length (T).

Figure 14 explores the impact of M and T on the relative cost. In the left panel, we see that for $s = 1$, ORC is less costly than FIX when the number of events is below 60. However, as s increases to 8, FIX becomes less costly when at most 10 events are present. These results indicate that the relative cost depends heavily on the density of sound events in the recording and the estimated annotation budget for ORC.

In the right panel, varying T shows a similar trend. For shorter recordings (high event density), ORC loses its cost advantage. However, it's important to note that the maximum achievable expected label accuracy with FIX under default settings ($\gamma = 0.5$) is $f^*(0.5) \approx 0.76$, whereas ORC achieves 1.0. In such cases, the additional cost of ORC may be justified by the significantly higher label quality.

While these results indicate that the relative cost depends on the sound event density, we should remember that we are considering weak labeling of presence events. This implies that all M events in this analysis are treated as non-overlapping, as the annotation task does not consider temporal overlaps for this analysis. The scenario of $M > 60$ non-overlapping events of length 1 in a recording of length $T = 100$ is therefore unlikely in practice. Similarly, estimating 10 events as 80 (modeled by $s = 8$) for an audio recording of length $T = 100$ represents a substantial overestimation and seems improbable given the capabilities of modern sound event detection tools.

7 Related Work

This work introduces a framework for characterizing segmentation label noise in FIX weak labeling, a largely unexplored area. Below, we review studies addressing noisy labels and approaches to mitigate their effects, with a focus on weak labeling in audio and related domains.

Dataset	Task	Fixed Length
CHIME (Foster et al., 2015)	Single-pass multi-label	4 seconds
AudioSet (Gemmeke et al., 2017)	Single-pass multi-label	10 seconds
MAESTRO Real (Martin-Morato & Mesaros, 2023)	Single-pass multi-label	10 seconds
OpenMIC-2018 (Humphrey et al., 2018)	Multi-pass binary-label	10 seconds

Table 1: Large-scale audio datasets using variations of FIX weak labeling.

7.1 Understanding Noisy Labels

Noisy labels are a partial description of the target model, influencing its performance. Early work by Liang et al. (2009) introduced the concept of *measurements* for conditional exponential families, encompassing labels and constraints for model learning with minimal human input—a goal shared by this work.

In deep learning, the ability of models to overfit noisy labels has prompted studies into the relationship between noise rate and generalization (Zhang et al., 2021; Chen et al., 2019). Research on class label noise often assumes a noise transition matrix (Li et al., 2021) but rarely considers spatially correlated errors like those arising in segmentation tasks (Yao et al., 2023). For audio, Hershey et al. (2021) demonstrated that training on strongly labeled data yields better results than weakly labeled data, highlighting the need for precise labels, particularly in evaluation.

7.2 Mitigating Noisy Labels

Several strategies address noisy labels, including regularization techniques like dropout (Srivastava et al., 2014), data augmentation (Shorten & Khoshgoftaar, 2019), and specialized loss functions (Fonseca et al., 2019). For weakly labeled audio, Dinkel et al. (2022) proposed a pseudo-labeling approach, iteratively refining labels to improve training performance. Despite these advances, most methods focus on training labels and offer limited insights into noisy evaluation labels, underscoring the need for frameworks that quantify label noise, such as the one proposed in this work.

7.3 Strong vs. Weak Labeling

Strong labeling, where the annotator provides the event boundaries and the class label, while often precise, is resource-intensive and subject to annotator variability (Mesaros et al., 2017). In bioacoustics, experts use spectrograms for efficient annotation (Cartwright et al., 2017), but the reliance on specialists limits scalability. Weak labeling, by contrast, simplifies the annotation task, which is especially important for crowd-sourced annotations, enabling broader data collection (Martin-Morato & Mesaros, 2023). However, segment label noise, especially at event boundaries, remains a significant challenge.

Large-scale audio datasets employing FIX weak labeling are summarized in Table 1. Two common annotation tasks are single-pass multi-label and multi-pass binary-label annotation (Cartwright et al., 2019). Single-pass multi-label annotation asks annotators to recognize the presence of multiple event classes during a single pass through the data. In contrast, multi-pass binary-label annotation asks annotators to detect the presence or absence of a single event class at a time through multiple passes through the data.

Cartwright et al. (2019) studied the trade-offs between these tasks and found that binary labeling is preferable when high recall is required. For example, AudioSet (Gemmeke et al., 2017) employs single-pass multi-label annotation with non-overlapping 10-second segments, which limits temporal resolution. Conversely, MAESTRO Real (Martin-Morato & Mesaros, 2023) uses overlapping 10-second segments with a 9-second overlap, increasing the accuracy of the derived labels.

The choice of segment length and overlap significantly impacts the utility of weak labeling. For example, while overlapping segments increase label accuracy (Martin-Morato & Mesaros, 2023), they still fail to distinguish events occurring close in time. Current work aims to better understand the effect of different choices of the segment length for FIX weak labeling.

7.4 Contributions of This Work

Existing research focuses predominantly on class label noise or assumes noise independence. This work extends these efforts by characterizing segment label noise specific to FIX weak labeling, providing a foundation for improving both training and evaluation processes in weakly labeled datasets.

8 Discussion

FIX labeling has been employed in many works, with varying degrees of complexity. Theorem 2 provides a useful rule of thumb for selecting the best segmentation length for a given event length, and Eq. 13 provides a way to use this theorem to analyze stochastic event length distributions. Our results suggest that, in most cases, knowing the average event length provides a good estimate, but understanding the (approximate) distribution of event lengths improves the analysis.

Implications for practical annotation. The analysis highlights the trade-offs in label accuracy and annotation cost between FIX and ORC weak labeling. While FIX can be less costly under specific conditions (e.g., high event density), these conditions are unlikely to occur in real-world annotation tasks. Furthermore, even in cases where FIX is less costly, its significantly lower label accuracy ($f^*(0.5) \approx 0.76$ vs. 1.0 for ORC) can negate its cost advantage. ORC, on the other hand, guarantees higher accuracy at a potentially higher cost, which is sensitive to overestimation of B_{ORC}.

Given the rarity of extreme event densities and the importance of high-quality labels, ORC is likely the better theoretical choice for most annotation tasks. However, ORC weak labeling is not available in practice since it uses the true change points of the events. An interesting research direction is to model the ORC weak labeling process by estimating these change points, construct query segments from these, and then let the annotator weakly label these query segments. Martinsson et al. (2024) propose to actively model the ORC weak labeling process during the annotation for sound event detection, and Kim et al. (2023) propose a related framework for image segmentation. However, how to model ORC weak labeling reliably in practice remains an open research question, as it may introduce unwanted bias due to annotation errors, overfitting to sparse events, or limitations in budget estimation models. In this regard, FIX weak labeling is very robust and provides a straightforward baseline that prioritizes simplicity and consistency.

Future research should focus on mitigating the potential biases when modeling ORC weak labeling while retaining its theoretical advantages. By addressing these challenges, annotation methods can better align with the practical needs of sound event detection and related applications.

Understanding the consequences of evaluating with noisy labels. Despite the extensive focus on noisy training labels, evaluation labels are often implicitly assumed to be perfect. However, as emphasized in the introduction, inaccurate evaluation labels present a significant challenge. Crucially, when noise is present in both training and evaluation data, we risk selecting models that merely replicate the evaluation noise, potentially overlooking those with superior generalization abilities. This echoes the central concern highlighted by Görnitz et al. (2014), which will be described below. To illustrate this point concretely, consider the implications of evaluating with noisy labels as described by our theorem.

We can use Theorem 3 to understand the properties of the best performing model when the evaluation data contains FIX weak labels. For example, we now know from Theorem 3 that for $\gamma = 0.5$ the annotations will at most have an expected label accuracy of $f^*(0.5) \approx 0.76$. The "best" performing model will therefore be a model that mimics the noise in these labels. We may end up rejecting a model that has learned the clean (accurate) labels simply because our evaluation labels are inaccurate. The theory therefore provides a better understanding for the "best" performing model under FIX weak labels.

$f^*(\gamma)$ as an upper bound. The expression for expected label accuracy derived in this paper applies to the simplest scenario, where only a single event with deterministic length is present. In all of our results, we observe that $f^*(\gamma)$ is greater than or equal to the expected and average label accuracy that FIX weak labeling achieve for more complex distributions. This suggests that $f^*(\gamma)$ can be considered an upper bound. However, a formal proof showing that adding more events or introducing event length variability leads to a harder distribution to annotate is beyond the scope of this paper.

Extending the theory to more dimensions. The theory we have derived is for data annotation in one dimension, in particular we exemplify with events in time for audio data. However, the framework can be extended to more dimensions. Annotation in more than three dimensions is hard, since human intuition starts to break down and designing annotation interfaces will be hard, so for most practical applications extending this theory to events that occur in three dimensions should be enough. With minor adjustments, this framework can be applied to FIX weak labeling of rectangles in images or cubes in point clouds, expanding its relevance to broader areas. This extension could open up new avenues for research in multi-dimensional label noise and help optimize annotation strategies in these domains.

Multiple different presence classes. If the same presence criterion γ is applicable for all classes then Theorem 1 is applicable to the joint event length distribution of these different classes. The integration performed in Eq. 13 can be adapted for the joint distribution, assuming it can be estimated. However, real-world presence criteria for different event classes may vary, requiring more complex models to account for these discrepancies. Future empirical studies on annotator behavior could help refine this model and improve its practical applicability.

Understanding segment-based evaluation criteria. The theory presented here is also relevant for evaluating sound event detection (SED) methods. Common evaluation metrics, such as the segment-based F_1 score (Mesaros et al., 2016), divide audio into fixed-length segments and label them based on overlap with ground truth. A key reason for using segment-based evaluation is that ground truth annotations are often temporally noisy or imprecise, and evaluating over longer segments helps to mitigate the impact of this temporal uncertainty. Assuming ground truth labels for evaluation, we effectively have an annotator that can detect the presence of arbitrarily small event fractions ($\gamma \to 0$). The expected label accuracy becomes $f(d_q) = d_e/(d_e + d_q)$, where d_q is the segment length. This formula suggests that a small d_q (approaching zero) leads to minimal segment label noise, but choosing a very small segment length negates the noise reduction effect and also comes at a computational cost. The theory presented here can help inform such trade-offs.

9 Conclusions

This study introduces a novel theoretical framework for understanding the trade-offs between label accuracy and annotation cost in weak labeling methods, particularly focusing on sound event detection where weak labeling is often employed to reduce annotation costs. We specifically compared fixed-length (FIX) and oracle (ORC) approaches.

We have demonstrated that FIX weak labeling, while cost-effective in specific scenarios, is inherently limited by segment label noise. The expressions we derived theoretically provide actionable insights into optimizing segment length for maximizing expected label accuracy under FIX. However, these results also underscore the fundamental trade-offs: shorter segments improve alignment with event boundaries but significantly increase annotation cost, while longer segments reduce cost at the expense of accuracy. In addition, how short these segments can be chosen depends on the ability of the annotator to detect presence of fractions of the events. In contrast, ORC labeling achieves perfect accuracy but can incur higher costs if events are very dense and the number of events are overestimated.

Our findings have several practical implications:

- **Annotation Strategy:** FIX weak labeling remains a robust, scalable choice for many practical applications. However, when high label accuracy is essential, ORC weak labeling—or adaptive methods approximating it—should be prioritized.
- **Adaptive Techniques:** Theoretical justification for adaptive weak labeling methods, e.g., methods based on active learning or iterative refinement, that mimic ORC weak labeling, which suggests promising avenues for improving annotation efficiency without compromising accuracy.
- **Evaluation Criteria:** Our analysis highlights the potential biases introduced by segment-level label noise in evaluating sound event detection models. Therefore, carefully aligning evaluation criteria with the intended model properties is critical.

Future research should address several limitations and extensions identified in our study. Developing practical approaches that reliably mimic ORC weak labeling by estimating the query segments without introducing a lot of unwanted bias in the labels remains an open challenge. Additionally, extending this framework to multi-dimensional data and multiple presence classes could broaden its applicability to other domains, such as medical imaging and point clouds.

In conclusion, the insights presented in this work offer a foundation for optimizing weak labeling processes, balancing cost and accuracy to meet the needs of diverse machine learning applications. By refining annotation strategies and leveraging adaptive methods, researchers can enhance the quality of labeled datasets. This, in turn, will drive advancements in supervised learning across domains, building upon the foundational understanding presented in this work.

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A Appendix

We do not include all simplifications of expressions in the proofs, but we do provide the code for a symbolic mathematics solver (SymPy) at GitHub[1], where all results can be verified. The notebook named "symbolic_verification_of_analysis.ipynb" can be used to verify the analysis.

A.1 Proof of Theorem 1

We will derive an expression for the expected query segment accuracy given overlap with a single event in terms of d_e, d_q, and γ, under all possible assumptions which will prove Theorem 1.

Proof. We need to consider two main assumptions. The first assumption is that the presence criterion for the annotator can be fulfilled, that is, $d_q \geq \gamma d_e$, and the second assumption is that the annotator presence criterion can not be fulfilled, that is, $d_q < \gamma d_e$. This happens if the query segment length is so short that it can never cover a large enough fraction of the event of interest to make presence detection feasible.

Assumption 1. The annotator presence criterion can be fulfilled ($d_q \geq \gamma d_e$).

Under this assumption there are two possible cases for the relation between d_q and d_e, either the event length is longer or equal to the query segment length, $d_e \geq d_q$ (case i), or the event length is shorter than the query segment length, $d_e < d_q$ (case ii). In Figure 15, we plot the query segment accuracy, $F(e_t, q, \gamma)$, for $t \in [0, d_e + d_q]$ for case (i) on the left, and case (ii) on the right. We describe in more detail in Appendix A.1.1 how the query segment accuracy behaves as a function of different amounts of overlap between the query segment and the event. Briefly, what we see in Figure 15 is that initially there is arbitrarily little overlap ($t_0^{(i)}$ and $t_0^{(ii)}$), an absence label is given to the query segment and the accuracy is therefore 1. Then the accuracy decrease linearly with the amount of overlap until the presence criterion is fulfilled and a presence label is given ($t_1^{(i)}$ and $t_1^{(ii)}$). After that, the accuracy linearly increase with the amount of overlap between the event and query segment until we reach a ceiling for the accuracy when either the whole query segment is inside the event ($t_2^{(i)}$) or the query segment covers the whole event ($t_2^{(ii)}$). Finally, the overlap between the query segment and the event starts to decrease again ($t_3^{(i)}$ and $t_3^{(ii)}$), and everything is symmetrical.

We continue by dropping the case superscripts show in the figure for $A_1, \ldots, A_3$ and $t_0, \ldots, t_5$, and only provide the full proof for case (i), but the proof for case (ii) is similar. In both cases the area A in Eq. 8 can be divided into five distinct parts:

$$A = 2A_1 + 2A_2 + A_3, \tag{21}$$

[1]link will be added for camera-ready, for now see the supplementary material

where A_1 and A_2 are counted twice due to symmetry.

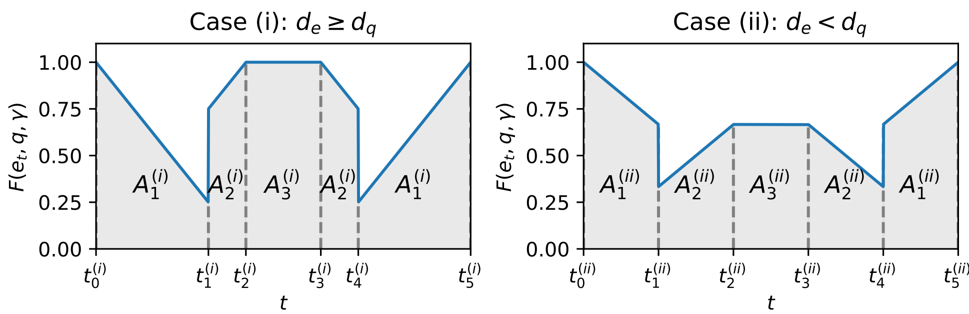

Figure 15: Assuming $d_q \geq d_e\gamma$, we plot the query segment accuracy, $F(e_t, q, \gamma)$, for $t \in [0, d_e + d_q]$, where $t_0 = 0$ and $t_5 = d_e + d_q$. Case (i) where $d_e \geq d_q$ is shown in the left panel, and case (ii) where $d_e < d_q$ is shown in the right panel.

The variables $t_0, t_1, \ldots, t_5$, represent the different states t of overlap where the discontinuities of $F(e_t, q, \gamma)$ occur, and using these we can express the areas as the following integrals:

$$A_1 = \int_{t_0}^{t_1} F(e_t, q, \gamma)\mathrm{d}t = \int_{t_4}^{t_5} F(e_t, q, \gamma)\mathrm{d}t, \tag{22}$$

and

$$A_2 = \int_{t_1}^{t_2} F(e_t, q, \gamma)\mathrm{d}t = \int_{t_3}^{t_4} F(e_t, q, \gamma)\mathrm{d}t, \tag{23}$$

due to symmetry, and

$$A_3 = \int_{t_2}^{t_3} F(e_q, q, \gamma)\mathrm{d}t. \tag{24}$$

We use that the query segment accuracy $F(e_t, q, \gamma)$ is linear in each interval, which means that the areas can be expressed as

$$A_1 = \frac{F(e_{t_0}, q, \gamma) + F(e_{t_{1-}}, q, \gamma)}{2}(t_1 - t_0), \tag{25}$$

$$A_2 = \frac{F(e_{t_{1+}}, q, \gamma) + F(e_{t_2}, q, \gamma)}{2}(t_2 - t_1), \tag{26}$$

and

$$A_3 = \frac{F(e_{t_2}, q, \gamma) + F(e_{t_3}, q, \gamma)}{2}(t_3 - t_2), \tag{27}$$

where t^- indicate that we approach the discontinuity at t from below and t^+ from above. We now only need to express $t_0, \ldots, t_3$ and $F(e_{t_0}, q, \gamma), \ldots, F(e_{t_3}, q, \gamma)$ in terms of d_e, d_q and γ to conclude the proof. For brevity, these have been provided in Table 2. See section A.1.1 for details on how to express these in terms of d_q, d_e and γ.

We provide the steps for case (i), and leave the derivation for case (ii) to the reader. We substitute the expressions for case (i), provided in Table 2, into equations Eq. 25-27, and the resulting expressions for the areas $A_1^{(i)}$, $A_2^{(i)}$, and $A_3^{(i)}$ into Eq. 21 which give

Case (i), $d_e \geq d_q$		Case (ii), $d_e < d_q$	
$t_0^{(i)} = 0$	$F(e_{t_0}^{(i)}, q, \gamma) = 1$	$t_0^{(ii)} = 0$	$F(e_{t_0}^{(ii)}, q, \gamma) = 1$
$t_1^{(i)} = \gamma d_e$	$F(e_{t_1^-}^{(i)}, q, \gamma) = \frac{d_q - \gamma d_e}{d_q}$	$t_1^{(ii)} = \gamma d_e$	$F(e_{t_1^-}^{(ii)}, q, \gamma) = \frac{d_q - \gamma d_e}{d_q}$
$t_2^{(i)} = d_q$	$F(e_{t_1^+}^{(i)}, q, \gamma) = \frac{\gamma d_e}{d_q}$	$t_2^{(ii)} = d_e$	$F(e_{t_1^+}^{(ii)}, q, \gamma) = \frac{\gamma d_e}{d_q}$
$t_3^{(i)} = d_e$	$F(e_{t_2}^{(i)}, q, \gamma) = 1$	$t_3^{(ii)} = d_q$	$F(e_{t_2}^{(ii)}, q, \gamma) = \frac{d_e}{d_q}$
	$F(e_{t_3}^{(i)}, q, \gamma) = 1$		$F(e_{t_3}^{(ii)}, q, \gamma) = \frac{d_e}{d_q}$

Table 2: A summary of the derived expressions for $t_0, \ldots, t_3$ and $F(e_{t_0}, q, \gamma), \ldots, F(e_{t_3}, q, \gamma)$ for each case. $F(e_{t_1^-}, q, \gamma)$ and $F(e_{t_1^+}, q, \gamma)$ denotes the limits when approaching t_1 from below and above respectively.

$$\begin{aligned} A^{(i)} &= \frac{2}{2}(1 + \frac{d_q - \gamma d_e}{d_q})\gamma d_e + \frac{2}{2}(1 + \frac{\gamma d_e}{d_q})(d_q - \gamma d_e) + (d_e - d_q) \\ &= (2d_q - \gamma d_e)\frac{\gamma d_e}{d_q} + (d_q + \gamma d_e)(d_q - \gamma d_e)\frac{1}{d_q} + (d_e - d_q) \\ &= \frac{1}{d_q}(2\gamma d_q d_e - \gamma^2 d_e^2 + \cancel{d_q^2} - \gamma^2 d_e^2 + d_e d_q - \cancel{d_q^2}) \\ &= \frac{1}{d_q}(2\gamma d_q d_e - 2\gamma^2 d_e^2 + d_e d_q) \\ &= \frac{d_e}{d_q}(2\gamma d_q - 2\gamma^2 d_e + d_q). \end{aligned}$$

Finally, by substituting A for $A^{(i)}$ in Eq. 8 we arrive at

$$\frac{A^{(i)}}{d_e + d_q} = \frac{d_e(2\gamma d_q - 2\gamma^2 d_e + d_q)}{d_q(d_e + d_q)} \tag{28}$$

which shows that Eq. 9 holds for case (i) under the assumption that $d_q \geq \gamma d_e$. Similarly, this also holds for case (ii).

Assumption 2. The annotator presence criterion can not be fulfilled ($d_q < \gamma d_e$).

When the presence criterion can not be fulfilled we never get any presence labels, this means that the fraction of the query segment that overlaps with an event is always incorrectly given an absence label. When the query segment completely overlaps with an event the query segment accuracy will be 0 (seen between t_1 and t_2 in Figure 16).

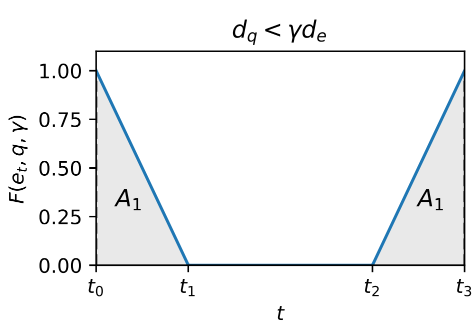

Figure 16: Assuming that $d_q < \gamma d_e$, we plot the query segment accuracy, $F(e_t, q, \gamma)$, for $t \in [0, d_e + d_q]$, where $t_0 = 0$ and $t_3 = d_e + d_q$.

The area A_1 is counted twice due to symmetry. The discontinuity at t_1 occurs for the smallest $t \in [0, d_e + d_q]$ for which $F(e_t, q, \gamma) = 0$, which happens for the smallest t for which the whole query segment overlaps with the event $|e \cap q| = d_q$ at $t = d_q$. We therefore have that $t_1 - t_0 = t_3 - t_2 = d_q$.

When there is no overlap between the query segment and the event giving a presence label is always correct, thus $F(e_{t_0}, q, \gamma) = 1$. However, giving an absence label to a query segment that completely overlaps with an event gives the query segment accuracy 0, thus $F(e_{t_1}, q, \gamma) = 0$. The total area under the curve is therefore $2A_1 = d_q$ and by normalizing with $t_3 - t_0 = d_e + d_q$, we get $d_q/(d_e + d_q)$, which proves the $d_q < \gamma d_e$ case of Eq. 9, and concludes the proof.

□

A.1.1 Details on the expressions in Table 2

This section provides a detailed explanation of the values presented in Table 2. For each case (i) and (ii), we will define the specific time points t_0, t_1, t_2, t_3 where the query segment accuracy function $F(e_t, q, \gamma)$ changes, and explain the corresponding value of the function at these points based on the overlap between the event e_t and the query segment q. The states t_4 and t_5 are analogous to t_1 and t_0, respectively, and therefore not illustrated. The difference is that the amount of overlap between the query segment and event decreases (instead of increases) when approaching these states.

Case (i): $d_e \geq d_q$

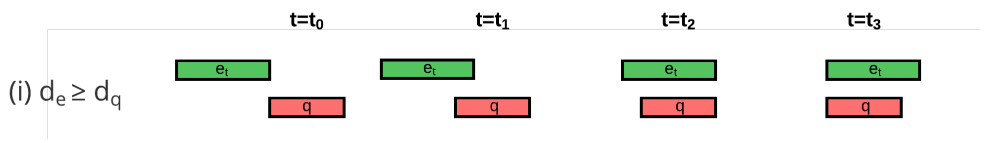

Figure 17: An illustration of how the sound event e_t and the query segment q overlap at the four distinct states $t = t_0, \ldots, t_3$ for case (i) where $d_e \geq d_q$.

- $t_0^{(i)}$: At $t_0^{(i)} = 0$, the end of the event e_t aligns perfectly with the beginning of the query segment q. This means there is no overlap between the event and the query segment ($|e_{t_0^{(i)}} \cap q| = 0$). Therefore, assuming the annotator absence criterion applies, the query segment accuracy is $F(e_{t_0^{(i)}}, q, \gamma) = \frac{d_q - |e_{t_0^{(i)}} \cap q|}{d_q} = \frac{d_q - 0}{d_q} = 1$.

- $t_1^{(i)}$: The time $t_1^{(i)} = \gamma d_e$ represents the point where the annotator presence criterion is first met. Before this point ($t < t_1^{(i)}$), the overlap $|e_t \cap q|$ is less than γd_e, and the query segment accuracy is given by $F(e_t, q, \gamma) = \frac{d_q - |e_t \cap q|}{d_q}$. As t approaches $t_1^{(i)}$ from the left, $|e_t \cap q|$ approaches γd_e, hence $\lim_{t \to t_1^-} F(e_t, q, \gamma) = \frac{d_q - \gamma d_e}{d_q}$. At $t = t_1^{(i)}$, the presence criterion is met, and the accuracy function switches to $F(e_t, q, \gamma) = \frac{|e_t \cap q|}{d_q}$. As t approaches $t_1^{(i)}$ from the right, $|e_t \cap q|$ is slightly greater than γd_e, and $\lim_{t \to t_1^+} F(e_t, q, \gamma) = \frac{\gamma d_e}{d_q}$. This transition is visually represented in Figure 17 at time $t = t_1$.

- $t_2^{(i)}$: At $t_2^{(i)} = d_q$, the entire query segment q is fully contained within the event e_t. This means the overlap is maximal: $|e_{t_2^{(i)}} \cap q| = d_q$. Since the presence criterion is met, the query segment accuracy is $F(e_{t_2^{(i)}}, q, \gamma) = \frac{|e_{t_2^{(i)}} \cap q|}{d_q} = \frac{d_q}{d_q} = 1$. This behavior is visually represented in Figure 17 at time $t = t_2$, where the green box representing the event fully covers the red box representing the query segment.

- $t_3^{(i)}$: At $t_3^{(i)} = d_e$, the entire query segment q still fully overlaps with the event e_t. Similar to t_2, the overlap is $|e_{t_3^{(i)}} \cap q| = d_q$, and therefore $F(e_{t_3^{(i)}}, q, \gamma) = \frac{|e_{t_3^{(i)}} \cap q|}{d_q} = \frac{d_q}{d_q} = 1$. This is depicted in Figure 17 at time $t = t_3$.

Case (ii): $d_e < d_q$

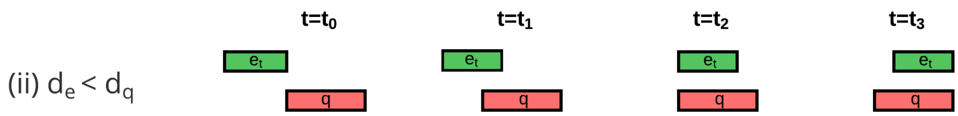

Figure 18: An illustration of how the sound event e_t and the query segment q overlap at the four distinct states $t = t_0, \ldots, t_3$ for case (ii) where $d_e < d_q$.

- $t_0^{(ii)}$: At $t_0^{(ii)} = 0$, the end of the event e_t aligns perfectly with the beginning of the query segment q. There is no overlap ($|e_{t_0^{(ii)}} \cap q| = 0$). Assuming the annotator absence criterion applies, the query segment accuracy is $F(e_{t_0^{(ii)}}, q, \gamma) = \frac{d_q - |e_{t_0^{(ii)}} \cap q|}{d_q} = \frac{d_q - 0}{d_q} = 1$.
- $t_1^{(ii)}$: The time $t_1^{(ii)} = \gamma d_e$ again marks the point where the annotator presence criterion is first met. Before this ($t < t_1^{(ii)}$), the overlap $|e_t \cap q| < \gamma d_e$, and $F(e_t, q, \gamma) = \frac{d_q - |e_t \cap q|}{d_q}$. Approaching $t_1^{(ii)}$ from the left, $|e_t \cap q| \to \gamma d_e$, thus $\lim_{t \to t_1^-} F(e_t, q, \gamma) = \frac{d_q - \gamma d_e}{d_q}$. At $t = t_1^{(ii)}$, the criterion is met, and the function becomes $F(e_t, q, \gamma) = \frac{|e_t \cap q|}{d_q}$. Approaching from the right, $|e_t \cap q|$ is slightly greater than γd_e, so $\lim_{t \to t_1^+} F(e_t, q, \gamma) = \frac{\gamma d_e}{d_q}$. This transition is shown in Figure 18 at $t = t_1$.
- $t_2^{(ii)}$: At $t_2^{(ii)} = d_e$, the beginning of the event e_t aligns with the beginning of the query segment q. At this point, the overlap is maximal, as the entire event is contained within the query segment: $|e_{t_2^{(ii)}} \cap q| = d_e$. Since the presence criterion is met, the query segment accuracy is $F(e_{t_2^{(ii)}}, q, \gamma) = \frac{|e_{t_2^{(ii)}} \cap q|}{d_q} = \frac{d_e}{d_q}$. This situation is illustrated in Figure 18 at $t = t_2$.
- $t_3^{(ii)}$: At $t_3^{(ii)} = d_q$, the end of the event e_t aligns with the end of the query segment q. Similar to $t_2^{(ii)}$, the entire event is contained within the query segment, so the overlap is $|e_{t_3^{(ii)}} \cap q| = d_e$. Consequently, the query segment accuracy is $F(e_{t_3^{(ii)}}, q, \gamma) = \frac{|e_{t_3^{(ii)}} \cap q|}{d_q} = \frac{d_e}{d_q}$. This corresponds to the state depicted in Figure 18 at $t = t_3$.

Understanding these key time points and the corresponding query segment accuracy values is crucial for calculating the area under the curve, which represents the expected query segment accuracy.

A.2 Proof of Theorem 2

Proof. We start by finding a unique critical point d_q^* which makes $f'(d_q^*) = 0$ when $d_q \geq \gamma d_e$. We then show that d_q^* is a global maximum by analyzing the boundaries of $f(d_q)$ on its' domain when $d_q \geq \gamma d_e$. We show that $f(d_q^*) \geq f(\gamma d_e)$ and that $f(d_q^*) \geq \lim_{d_q \to \infty} f(d_q)$. Since d_q^* is a unique critical point we conclude that it must be a global maximum of the function $f(d_q)$ when $d_q \geq \gamma d_e$. Lastly, we show that $f(d_q^*) \geq f(\gamma d_e) \geq f(d_q)$ when $d_q < \gamma d_e$ which proves that d_q^* is a global maximum of the function $f(d_q)$ for $d_q > 0$.

1. Finding the unique critical point d_q^*.

To find the critical points, we need to compute the derivative of $f(d_q)$ with respect to d_q and set it to zero. Let $N(d_q) = d_e(-2d_e\gamma^2 + 2d_q\gamma + d_q)$ and $D(d_q) = d_q(d_e + d_q)$. Then $f(d_q) = \frac{N(d_q)}{D(d_q)}$. Using the quotient rule, the derivative is given by:

$$f'(d_q) = \frac{N'(d_q)D(d_q) - N(d_q)D'(d_q)}{[D(d_q)]^2}$$

First, we find the derivatives of the numerator and the denominator:

$$\begin{aligned} N'(d_q) &= \frac{d}{dd_q}[d_e(-2d_e\gamma^2 + 2d_q\gamma + d_q)] \\ &= d_e(0 + 2\gamma + 1) \\ &= d_e(2\gamma + 1) \end{aligned}$$

$$\begin{aligned} D(d_q) &= d_q(d_e + d_q) = d_e d_q + d_q^2 \\ D'(d_q) &= \frac{d}{dd_q}[d_e d_q + d_q^2] \\ &= d_e + 2d_q \end{aligned}$$

Now, we plug these into the quotient rule formula:

$$f'(d_q) = \frac{[d_e(2\gamma+1)][d_q(d_e+d_q)] - [d_e(-2d_e\gamma^2 + 2d_q\gamma + d_q)][d_e + 2d_q]}{[d_q(d_e+d_q)]^2}$$

To find the critical points, we set $f'(d_q) = 0$, which means the numerator must be zero:

$$[d_e(2\gamma+1)][d_q(d_e+d_q)] - [d_e(-2d_e\gamma^2 + 2d_q\gamma + d_q)][d_e + 2d_q] = 0$$

Since $d_e > 0$, we can divide by d_e:

$$(2\gamma+1)d_q(d_e+d_q) - (-2d_e\gamma^2 + 2d_q\gamma + d_q)(d_e + 2d_q) = 0$$

Expanding the terms:

$$\begin{aligned} (2\gamma+1)(d_e d_q + d_q^2) - (-2d_e^2\gamma^2 - 4d_e d_q\gamma^2 + 2d_e d_q\gamma + 4d_q^2\gamma + d_e d_q + 2d_q^2) &= 0 \\ 2\gamma d_e d_q + 2\gamma d_q^2 + d_e d_q + d_q^2 - (-2d_e^2\gamma^2 - 4d_e d_q\gamma^2 + 2d_e d_q\gamma + 4d_q^2\gamma + d_e d_q + 2d_q^2) &= 0 \end{aligned}$$

Collecting and rearranging the terms to form a quadratic equation in d_q:

$$\begin{aligned} (2\gamma + 1 - 4\gamma - 2)d_q^2 + (2\gamma + 1 + 4\gamma^2 - 2\gamma - 1)d_e d_q + 2d_e^2\gamma^2 &= 0 \\ (-2\gamma - 1)d_q^2 + (4\gamma^2)d_e d_q + 2d_e^2\gamma^2 &= 0 \\ (2\gamma + 1)d_q^2 - 4\gamma^2 d_e d_q - 2d_e^2\gamma^2 &= 0 \end{aligned}$$

Using the quadratic formula $d_q = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $a = 2\gamma + 1$, $b = -4d_e\gamma^2$, $c = -2d_e^2\gamma^2$:

$$\begin{aligned} d_q &= \frac{4d_e\gamma^2 \pm \sqrt{(-4d_e\gamma^2)^2 - 4(2\gamma+1)(-2d_e^2\gamma^2)}}{2(2\gamma+1)} \\ &= \frac{4d_e\gamma^2 \pm \sqrt{16d_e^2\gamma^4 + 8(2\gamma+1)d_e^2\gamma^2}}{4\gamma+2} \\ &= \frac{4d_e\gamma^2 \pm \sqrt{16d_e^2\gamma^4 + 16d_e^2\gamma^3 + 8d_e^2\gamma^2}}{4\gamma+2} \\ &= \frac{4d_e\gamma^2 \pm \sqrt{8d_e^2\gamma^2(2\gamma^2 + 2\gamma + 1)}}{4\gamma+2} \\ &= \frac{4d_e\gamma^2 \pm 2d_e|\gamma|\sqrt{4\gamma^2 + 4\gamma + 2}}{2(2\gamma+1)} \end{aligned}$$

Since $\gamma > 0$, we have $|\gamma| = \gamma$:

$$\begin{aligned} d_q &= \frac{4d_e\gamma^2 \pm 2d_e\gamma\sqrt{4\gamma^2 + 4\gamma + 2}}{2(2\gamma+1)} \\ &= \frac{2d_e\gamma^2 \pm d_e\gamma\sqrt{4\gamma^2 + 4\gamma + 2}}{(2\gamma+1)} \\ &= d_e\gamma\frac{2\gamma \pm \sqrt{4\gamma^2 + 4\gamma + 2}}{2\gamma+1} \end{aligned}$$

We note that $\sqrt{4\gamma^2+4\gamma+2} = 2\sqrt{\gamma^2+\gamma+0.5} > 2\sqrt{\gamma^2} = 2\gamma$, which means that we need to choose the positive sign for $d_q > 0$ to be true. The value of d_q that makes the derivative zero is therefore uniquely defined by:

$$d_q = d_e\,\gamma \frac{2\,\gamma + \sqrt{4\,\gamma^2 + 4\,\gamma + 2}}{2\,\gamma + 1} \geq d_e\gamma,$$

where the last inequality holds because $\sqrt{4\gamma^2+4\gamma+2} = 2\sqrt{\gamma^2+\gamma+0.5} \geq 1$.

2. Analyze the function at the boundaries of its' domain.

To understand why this critical point corresponds to a maximum, we analyze the function $f(d_q)$ as d_q at the boundaries of its domain.

2a. $f(d_q)$ **when** $d_q = \gamma d_e$ ($d_q \geq \gamma d_e$).

$$\begin{aligned}
f(\gamma d_e) &= \frac{d_e\left(2(\gamma d_e)\gamma - 2d_e\gamma^2 + (\gamma d_e)\right)}{(\gamma d_e)\,(d_e + \gamma d_e)} \\
&= \frac{d_e\left(2d_e\gamma^2 - 2d_e\gamma^2 + \gamma d_e\right)}{(\gamma d_e)\,(d_e + \gamma d_e)} \\
&= \frac{d_e\,(\gamma d_e)}{(\gamma d_e)\,(d_e + \gamma d_e)} \\
&= \frac{d_e^2\gamma}{(\gamma d_e) d_e (1+\gamma)} \\
&= \frac{1}{1+\gamma}.
\end{aligned}$$

2b. $f(d_q)$ **as** $d_q \to \infty$ ($d_q \geq \gamma d_e$).

We want to evaluate the limit of $f(d_q)$ as d_q approaches infinity:

$$\lim_{d_q\to\infty} f(d_q) = \lim_{d_q\to\infty} \frac{d_e(-2d_e\gamma^2 + (2\gamma+1)d_q)}{d_e d_q + d_q^2}$$

Divide the numerator and the denominator by the highest power of d_q in the denominator, which is d_q^2:

$$\lim_{d_q\to\infty} f(d_q) = \lim_{d_q\to\infty} \frac{d_e\left(-\frac{2d_e\gamma^2}{d_q^2} + \frac{2\gamma+1}{d_q}\right)}{\frac{d_e}{d_q} + 1}$$

As $d_q \to \infty$, the terms $\frac{2d_e\gamma^2}{d_q^2}$, $\frac{2\gamma+1}{d_q}$, and $\frac{d_e}{d_q}$ all approach 0. Thus,

$$\lim_{d_q\to\infty} f(d_q) = \frac{d_e(0+0)}{0+1} = 0$$

This means that as d_q becomes very large, the function $f(d_q)$ approaches 0.

2c. Showing that $f(d_q^*) \geq f(\gamma d_e)$.

We want to show that $f(d_q^*) \geq f(\gamma d_e)$. Or equivalently, that $f(d_q^*) - f(\gamma d_e) \geq 0$. From Theorem 3 we know that $f(d_q^*) = 2\gamma\left(2\gamma + 1 - \sqrt{4\gamma^2+4\gamma+2}\right) + 1$, and from 2a we know that $f(\gamma d_e) = \frac{1}{1+\gamma}$. After substitution and some algebraic manipulation, we get

$$\gamma\left(\frac{4\gamma^2+6\gamma+3}{1+\gamma} - 2\sqrt{4\gamma^2+4\gamma+2}\right) \geq 0.$$

Since $\gamma > 0$, it suffices to show that

$$\frac{4\gamma^2+6\gamma+3}{1+\gamma} \geq 2\sqrt{4\gamma^2+4\gamma+2}.$$

Squaring both sides of the above inequality and simplifying, we obtain the equivalent inequality

$$\left(\frac{4\gamma^2+6\gamma+3}{1+\gamma}\right)^2 \geq 4(4\gamma^2+4\gamma+2).$$

After further algebraic manipulations (which we leave to the reader), we arrive at the inequality

$$(2\gamma+1)^2 \geq 0.$$

Since $(2\gamma+1)^2 \geq 0$ holds for all γ, and the previous steps are all equivalences, we conclude that

$$f(d_q^*) - f(\gamma d_e) \geq 0$$

for $0 < \gamma \leq 1$, and therefore,

$$f(d_q^*) \geq f(\gamma d_e).$$

2d. Showing that $f(d_q^*) \geq \lim_{d_q\to\infty} f(d_q)$.

We combine the results from 2a-2c to get

$$\begin{aligned} f(d_q^*) &\geq f(\gamma d_e) \\ &= \frac{1}{1+\gamma} \\ &\geq 0 \\ &= \lim_{d_q\to\infty} f(d_q). \end{aligned}$$

2e. $f(d_q)$ as $d_q \to (\gamma d_e)^-$ ($d_q < \gamma d_e$).

Since we are approaching γd_e from the left, we have that $f(d_q) = d_q/(d_e + d_q)$. This function is continuous for $d_q < \gamma d_e$, so the limit is given by the direct substitution:

$$\begin{aligned} \lim_{d_q\to(\gamma d_e)^-} \frac{d_q}{d_e+d_q} &= \frac{\gamma d_e}{d_e+\gamma d_e} \\ &= \frac{\gamma d_e}{d_e(1+\gamma)} \\ &= \frac{\gamma}{1+\gamma} \end{aligned}$$

2f. Showing that $f(\gamma d_e) \geq f(d_q)$ when $d_q < \gamma d_e$. We start by noting that $f(\gamma d_e) = \frac{1}{1+\gamma} \geq \frac{\gamma}{1+\gamma} = \lim_{d_q\to(\gamma d_e)^-}$. Now it is sufficient to show that $f(d_q) = d_q/(d_q + d_e)$ is strictly decreasing for decreasing d_q, which we do by computing the derivative of $f(d_q)$ with respect to d_q using the quotient rule:

$$\begin{aligned} f'(d_q) &= \frac{(d_q+\gamma)(1) - d_q(1)}{(d_q+\gamma)^2} \\ &= \frac{d_q+\gamma-d_q}{(d_q+\gamma)^2} \\ &= \frac{\gamma}{(d_q+\gamma)^2}. \end{aligned}$$

Since $\gamma > 0$ and $(d_q+\gamma)^2 > 0$ for all $d_q > 0$, we have $f'(d_q) > 0$ for all $d_q > 0$. This implies that the function $f(d_q)$ is strictly increasing on the interval $(0, \infty)$. Therefore, if $0 < c \leq b$, it must be the case that $f(c) \leq f(b)$. Moreover, since $c < b$, $f(c) < f(b)$. Thus, for any $b > 0$, $f(b) > f(c)$ for all $0 < c \leq b$. Now let $0 < d_q = c \leq \gamma d_e = b$.

3. Combining everything (2a-2f)

We have derived a unique critical point $d_q^* \geq \gamma d_e$ by setting the first derivative of $f(d_q)$ to zero. We have then shown that $f(d_q^*)$ is greater than or equal to $f(d_q)$ at the limits of its' domain when $d_q \geq \gamma d_e$. Finally, we show that $f(d_q^*) \geq f(\gamma d_e) \geq f(d_q)$ when $d_q < \gamma d_e$. Therefore, the value of d_q that is the global maximum of $f(d_q)$ when $d_q > 0$ is:

$$\boxed{d_q^* = d_e\,\gamma \frac{2\,\gamma + \sqrt{4\,\gamma^2 + 4\,\gamma + 2}}{2\,\gamma + 1}}$$

□

A.3 Proof of Theorem 3

Proof. From Theorem 2 we have that

$$d_q^* \;=\; \frac{d_e\,\gamma\left(2\,\gamma + \sqrt{4\,\gamma^2 + 4\,\gamma + 2}\right)}{2\,\gamma + 1}$$

maximizes the function

$$f(d_q) \;=\; \frac{d_e\left(-2\,d_e\,\gamma^2 \;+\; (2\,\gamma + 1)\,d_q\right)}{d_q\left(d_e + d_q\right)}.$$

We wish to show that the maximum label accuracy given overlap, $f^*(\gamma) = f\left(d_q^*\right)$, is

$$2\,\gamma\Big(2\,\gamma + 1 \;-\; \sqrt{4\,\gamma^2 + 4\,\gamma + 2}\Big) \;+\; 1.$$

1. Express $f(d_q)$ in terms of a dimensionless variable.

Define

$$\delta \;=\; \frac{d_q}{d_e}.$$

Then

$$d_q \;=\; \delta\,d_e, \quad d_e + d_q \;=\; d_e\,(1 + \delta),$$

and

$$f(d_q) \;=\; f(\delta\,d_e) \;=\; \frac{d_e\left(-2\,d_e\,\gamma^2 + (2\,\gamma + 1)\,\delta\,d_e\right)}{(\delta\,d_e)\left(d_e + \delta\,d_e\right)} = \frac{-2\,\gamma^2 + (2\,\gamma + 1)\,\delta}{\delta\left(1 + \delta\right)}.$$

We can therefore write

$$f(\delta) \;=\; \frac{-2\,\gamma^2 \;+\; (2\,\gamma + 1)\,\delta}{\delta\,(1 + \delta)}.$$

2. Identify the optimal dimensionless query length δ^*.

From Theorem 2, we know that

$$d_q^* \;=\; \frac{d_e\,\gamma\left(2\,\gamma \;+\; \sqrt{4\,\gamma^2 + 4\,\gamma + 2}\right)}{2\,\gamma + 1}.$$

Dividing both sides by d_e gives

$$\delta^* \;=\; \frac{d_q^*}{d_e} \;=\; \gamma\,\frac{2\,\gamma \;+\; \sqrt{4\,\gamma^2 + 4\,\gamma + 2}}{2\,\gamma + 1}.$$

We need to show that

$$f\left(\delta^*\right) \;=\; 2\,\gamma\Big(2\,\gamma + 1 - \sqrt{4\,\gamma^2 + 4\,\gamma + 2}\Big) \;+\; 1.$$

3. Compute $f(\delta^*)$ explicitly.

Let

$$N(\delta) \;=\; -2\,\gamma^2 \;+\; (2\,\gamma+1)\,\delta, \quad D(\delta) \;=\; \delta\,(1+\delta).$$

Then $f(\delta) = \frac{N(\delta)}{D(\delta)}$.

1. *Numerator at δ^*.*

$$N\big(\delta^*\big) = -2\,\gamma^2 \;+\; (2\,\gamma+1)\,\delta^* = -2\,\gamma^2 \;+\; (2\,\gamma+1)\Big[\gamma\,\frac{2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}}{2\,\gamma+1}\Big].$$

Inside the brackets, $(2\,\gamma+1)$ cancels:

$$N\big(\delta^*\big) = -2\,\gamma^2 + \gamma\,\big(2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}\big) = -2\,\gamma^2 + 2\,\gamma^2 + \gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2} = \gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2}.$$

2. *Denominator at δ^*.*

$$D(\delta) \;=\; \delta\,(1+\delta).$$

Hence,

$$D\big(\delta^*\big) = \delta^*\Big(1+\delta^*\Big) = \Big[\gamma\,\frac{2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}}{2\,\gamma+1}\Big]\Big[1 \;+\; \gamma\,\frac{2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}}{2\,\gamma+1}\Big].$$

The second bracket becomes a single fraction:

$$1+\gamma\,\frac{2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}}{2\,\gamma+1} = \frac{(2\,\gamma+1) \;+\; \gamma\,\big(2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}\big)}{2\,\gamma+1}.$$

Combining, we get

$$D\big(\delta^*\big) = \gamma\,\frac{2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}}{2\,\gamma+1} \;\times\; \frac{(2\,\gamma+1)+2\,\gamma^2+\gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2}}{2\,\gamma+1}.$$

So

$$D\big(\delta^*\big) = \gamma\,\frac{\big(2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}\big)\,\big(2\,\gamma+1+2\,\gamma^2+\gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2}\big)}{(2\,\gamma+1)^2}.$$

3. *Form the ratio.* Thus,

$$f\big(\delta^*\big) = \frac{N(\delta^*)}{D(\delta^*)} = \frac{\gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2}}{\gamma\,\frac{\big(2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}\big)\,\big(2\,\gamma+1+2\,\gamma^2+\gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2}\big)}{(2\,\gamma+1)^2}}.$$

Cancel the common factor γ, invert the denominator and multiply:

$$f\big(\delta^*\big) = \frac{\sqrt{4\,\gamma^2+4\,\gamma+2}\,(2\,\gamma+1)^2}{\big(2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}\big)\,\big(2\,\gamma+1+2\,\gamma^2+\gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2}\big)}.$$

You can verify by direct expansion (or by a symbolic algebra tool which we provide in the supplementary material) that

$$\frac{\sqrt{4\,\gamma^2+4\,\gamma+2}\,(2\,\gamma+1)^2}{\big(2\,\gamma+\sqrt{4\,\gamma^2+4\,\gamma+2}\big)\,\big(2\,\gamma+1+2\,\gamma^2+\gamma\,\sqrt{4\,\gamma^2+4\,\gamma+2}\big)} = 2\,\gamma\,\Big(2\,\gamma+1-\sqrt{4\,\gamma^2+4\,\gamma+2}\Big) \;+\; 1.$$

Thus

$$f\big(\delta^*\big) \;=\; 2\,\gamma\,\Big(2\,\gamma+1-\sqrt{4\,\gamma^2+4\,\gamma+2}\Big)+1,$$

which proves that

$$f^*(\gamma) = f\big(d_q^*\big) = 2\,\gamma\,\Big(2\,\gamma+1-\sqrt{4\,\gamma^2+4\,\gamma+2}\Big) \;+\; 1.$$

Hence, Eq. 11 holds, completing the proof. □